MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXX.

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EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF

THE GENERAL MEDICAL COUNCIL.

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Solved Questions.

CORRIGENDA.

VOL. XXIX.

p. 84, end of line 3 from bottom, *for* $r-x$ *read* $x-r$.

p. 84, last line, *for* $(\ldots)\, x\, dx$ *read* $(\ldots)^2\, x\, dx$.

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p. 24, line 7 from bottom, *for* $+a^2$ *read* $-a^2$.

p. 24, lines 2 and 6 from bottom, *for* $\sin^2 c$ *read* $\sin^2 C$.

p. 25, line 6, *for* $\sin c$ *read* $\sin C$.

p. 81, line 13, *for* 573 (No. of Quest.) *read* 5738.

p. 88, line 20, *read* " Quest. 5735 (see p. 62 of this Volume)."

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5628. (By Professor MINCHIN, M.A.)—Show that a system of forces acting on a rigid body can be astatically equilibrated by two forces when all the given forces are parallel to one plane. Also show that, in the general case, if each of the given forces is resolved into three components parallel to fixed rectangular directions, and if the centres of the three systems of parallel forces thus obtained be collinear, the given system may be astatically equilibrated by two forces; and that, if these latter are taken perpendicular to each other, their points of application are conjugate points of an involution system on the line of the above centres.

5663. (By Professor MINCHIN, M.A.)—When a system of forces applied to a body can be astatically equilibrated by two forces, show that, if these latter are assumed rectangular, they will, as their points of application vary, be represented in magnitudes and directions by the sides of a (variable) right-angled triangle described in a plane fixed in space, with the Resultant of Translation of the system as hypotenuse; and that their lines of action trace out a hyperbolic paraboloid.

Solution by J. J. WALKER, M.A.

If a system of forces applied at points $xyz\,...$, whose components are $X\,...\,Y\,...\,Z\,...$ in three rectangular directions, can be equilibrated astatically by two forces acting at points $x_1y_1z_1$, $x_2y_2z_2$, and whose components are $X_1Y_1Z_1$, $X_2Y_2Z_2$ respectively, the conditions

$$\Sigma X(\Sigma Yx\,.\,\Sigma Zy-\Sigma Zx\,.\,\Sigma Yy)+\Sigma Y(\Sigma Xy\,.\,\Sigma Zx-\Sigma Zy\,.\,\Sigma Xx)$$
and
$$+\,\Sigma Z(\Sigma Yy\,.\,\Sigma Xx-\Sigma Xy\,.\,\Sigma Yx)=0\,.......\,(1)$$

$$\Sigma X(\Sigma Yx\,.\,\Sigma Zz-\Sigma Zx\,.\,\Sigma Yz)+\Sigma Y(\Sigma Xz\,.\,\Sigma Zx-\Sigma Zz\,.\,\Sigma Xx)$$
$$+\,\Sigma Z(\Sigma Yz\,.\,\Sigma Xx-\Sigma Xz\,.\,\Sigma Yx)=0\,.......\,(2),$$

must be satisfied. These conditions may be proved exactly as the corresponding ones for a force and a couple are in MOIGNO's *Leçons de Mécanique.* If the components $Z\,...$ each vanish, these conditions are

obviously satisfied. For the second part of the question, let $X' = \Sigma X$, $Y' = \Sigma Y$, $Z' = \Sigma Z$, and let the centres of these parallel systems be $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ respectively. The three points being collinear,

$$\alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta'' + \alpha\beta' - \alpha'\beta = 0,$$

and $$\alpha'\gamma'' - \alpha''\gamma' + \alpha''\gamma - \alpha\gamma'' + \alpha\gamma' - \alpha'\gamma = 0\,;$$

and multiplying by $X'Y'Z'$, these equations become the same as (1), (2) in virtue of $\alpha'Y' = \Sigma Yx$, &c. Again,

$$X' + X_1 + X_2 = 0, \quad Y' + Y_1 + Y_2 = 0, \quad Z' + Z_1 + Z_2 = 0\,;$$

$$X'\alpha + X_1x_1 + X_2x_2 = 0, \quad \text{or} \quad X'(\alpha - x_1) = X_2(x_1 - x_2), \quad \text{and} \quad X'(\alpha - x_2) = X_1(x_2 - x_1)\,;$$

$$X'\beta + X_1y_1 + X_2y_2 = 0, \quad \text{or} \quad X'(\beta - y_1) = X_2(y_1 - y_2), \quad \text{and} \quad X'(\beta - y_2) = X_1(y_2 - y_1).$$

Hence $$(\alpha - x_1)(y_1 - y_2) - (\beta - y_1)(x_1 - x_2) = 0,$$

and similarly $$(\alpha - x_1)(z_1 - z_2) - (\gamma - z_1)(x_1 - x_2) = 0\,;$$

that is, the points of application of the two forces are collinear with the centres of the systems of components.

Also, $$X_1X_2(x_1 - x_2)^2 + X'^2(\alpha - x_1)(\alpha - x_2) = 0,$$

and similarly $$Y_1Y_2(x_1 - x_2)^2 + Y'^2(\alpha' - x_1)(\alpha' - x_2) = 0,$$

$$Z_1Z_2(x_1 - x_2)^2 + Z'^2(\alpha'' - x_1)(\alpha'' - x_2) = 0\,;$$

hence, if $$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0,$$

$$(X'^2 + Y'^2 + Z'^2)x_1x_2 + (\alpha X'^2 + \alpha'Y'^2 + \alpha''Z'^2)(x_1 + x_2) + \alpha^2X'^2 + \alpha'^2Y'^2 + \alpha''^2Z'^2 = 0,$$

which is the involution relation, if the axis of x be taken parallel to the line of centres.

Whether the two forces whose components are $X_1Y_1Z_1$, $X_2Y_2Z_2$ are perpendicular or not,

$$(Y_1Z_2 - Y_2Z_1)\Sigma Xx + (Z_1X_2 - Z_2X_1)\Sigma Yx + (X_1Y_2 - X_2Y_1)\Sigma Zx = 0,$$

$$(\;\ldots\quad\ldots\;)\Sigma Xy + (\;\ldots\quad\ldots\;)\Sigma Yy + (\;\ldots\quad\ldots\;)\Sigma Zy = 0,$$

$$(\;\ldots\quad\ldots\;)\Sigma Xz + (\;\ldots\quad\ldots\;)\Sigma Yz + (\;\ldots\quad\ldots\;)\Sigma Zz = 0,$$

[leading to $\Sigma Xx(\Sigma Yy \,.\, \Sigma Zz - \Sigma Yz \,.\, \Sigma Zz) + \Sigma Xy(\ldots) + \Sigma Xz(\ldots) = 0$, which follows from the two conditions given above]; consequently the normal to the plane parallel to each of the two forces, in any case, is fixed in direction. Also, for every pair of forces,

$$(Y_1Z_2 - Y_2Z_1)\Sigma X + (Z_1X_2 - Z_2X_1)\Sigma Y + (X_1Y_2 - X_2Y_1)\Sigma Z = 0\,;$$

so that the first statement in Question 5663 may be extended to every case of Astaticism by two forces—viz., that these forces and the Resultant of Translation are parallel to a fixed plane, and are proportional to the sides of a triangle parallel to their directions.

In the case of two rectangular forces, taking as axis of z the line of centres, and placing the origin at the centre of the components $Z\ldots$, the equations to the line of action of one force are

$$X_1y = Y_1x,$$

$$\{z(X'Y_1 - Y'X_1) - \gamma X(Y_1 - Y') + \gamma'Y'(X_1 - X')\}X_1 = x(X'Y_1 - Y'X_1)Z_1$$

with the conditions

$$Z'(\gamma'Y'X_1 - \gamma X'Y_1) + (\gamma - \gamma)X'Y'Z_1 = 0,$$

$$X_1^2 + Y_1^2 + Z_1^2 = X'X_1 + Y'Y_1 + Z'Z_1,$$

(where X'... stand for ΣX...). Eliminating $X_1Y_1Z_1$, the result, after division by $X'y-Y'x$, is

$$\{(\gamma-\gamma')X'Y'(X'x+Y'y)+Z'^2(\gamma X'y-\gamma'Y'x)\} \times \{(\gamma-\gamma')X'Y'z-Z'(\gamma X'y-\gamma'Y'x)\} = X'^2Y'^2(\gamma-\gamma')^2(\gamma X'x+\gamma'Y'y).$$

[Professor MINCHIN states that his results were obtained by *Quaternions*, which, he adds, "are much more simple and ready for the purpose than *Cartesian Crutches*."]

5419. (By R. TUCKER, M.A.)—Prove that

$$\frac{\pi^2}{16} = R(1)-\tfrac{1}{2}R(3)+\tfrac{1}{3}R(5)-\&c.-\frac{1}{2n}R(4n-1)+\frac{1}{2n+1}R(4n+1)-\&c.,$$

where $R(2p+1)$ stands for sum of reciprocals of odd numbers up to $2p+1$.

I. *Solution by* J. HAMMOND, M.A.; Prof. MOREL; *and others.*

The series may be written

$$1-\tfrac{1}{2}(1+\tfrac{1}{3})+\tfrac{1}{3}(1+\tfrac{1}{3}+\tfrac{1}{5})-\tfrac{1}{4}(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})+\&c.$$
$$=\int_0^1 dx\left\{1-x(1+\tfrac{1}{3})+x^2(1+\tfrac{1}{3}+\tfrac{1}{5})+x^3(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})+\&c.\right\}$$
$$=\int_0^1\frac{dx}{1+x}\left\{1-\frac{x}{3}+\frac{x^2}{5}-\frac{x^3}{7}+\dots\right\}.$$

Putting $x=\tan^2\theta$, this reduces to $\int_0^{\frac{1}{4}\pi}2\theta\,d\theta=\frac{\pi^2}{16}$.

II. *Solution by the* PROPOSER.

Let $t=\tan\theta$; then we have

$$\theta=t-\frac{t^3}{3}+\frac{t^5}{5}-\dots-\frac{t^{4n-1}}{4n-1}+\frac{t^{4n+1}}{4n+1}-\&c.;$$

$$\therefore\quad \theta^2=\dots-2t^{4n}\left[\frac{1}{4n-1}+\frac{1}{3.4n-3}+\dots\right]+t^{4n+2}\times\left[\frac{1}{(2n+1)^2}+2\left\{\frac{1}{4n+1}+\frac{1}{3.4n-1}+\dots\right\}\right]-\dots$$
$$=\dots-\frac{1}{2n}t^{4n}R(4n-1)+\frac{1}{2n+1}t^{4n+2}R(4n+1)-\dots.$$

Let $\theta=\frac{1}{4}\pi$; then we have

$$\tfrac{1}{16}\pi^2=R(1)-\tfrac{1}{2}R(3)+\dots-\frac{1}{2n}R(4n-1)+\frac{1}{2n+1}R(4n+1)-\dots.$$

4856. (By the EDITOR.)—Given the three axial foci of a Cartesian, prove that the locus of the points of contact of its double tangent is the conic $$y^2 = 3x^2 - 2(\alpha+\beta+\gamma)x + \beta\gamma + \gamma\alpha + \alpha\beta,$$ where α, β, γ are the distances of the foci from the origin. [See the Solution of Question 4778, *Reprint*, Vol. XXIV., p. 94.]

Solution by J. HAMMOND, M.A.

The α, β, γ of Question 4778 are the distances of the three foci from the triple focus, and must be replaced by $\alpha - t$, $\beta - t$, $\gamma - t$, where t is the distance of the triple focus from the origin.

The equation of the Cartesian is now

$$\left\{(x-t)^2 + y^2 - (\beta-t)(\gamma-t) - (\gamma-t)(\alpha-t) - (\alpha-t)(\beta-t)\right\}^2 + 4(\alpha-t)(\beta-t)(\gamma-t)(2x-\alpha-\beta-\gamma+t) = 0,$$

and the points of contact of the double tangent are given by

$$2x-\alpha-\beta-\gamma+t = 0, \quad x^2+y^2-2t(x-\alpha-\beta-\gamma+t)-\beta\gamma-\gamma\alpha-\alpha\beta = 0.$$

Substituting in the second of these the value of t in the first, we get

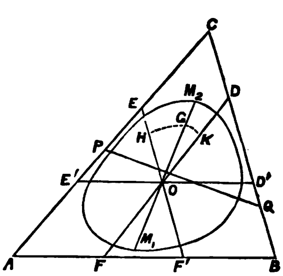
$$y^2 + x^2 + 2x(\alpha+\beta+\gamma-2x) - \beta\gamma - \gamma\alpha - \alpha\beta = 0,$$

or $$y^2 = 3x^2 - 2x(\alpha+\beta+\gamma) + \beta\gamma + \gamma\alpha + \alpha\beta,$$

the locus of the points of contact of the double tangent.

5333. (By Colonel A. R. CLARKE, C.B., F.R.S.)—A given weight is placed at random on a floating triangular lamina; find the chance that it is upset.

Solution by the PROPOSER.



Let i be the (infinitesimal) thickness of the lamina; h the depth to which it sinks when the weight w is placed over its centroid; W, Δ, σ the weight, area, and specific gravity of the lamina; and $W = nw$. Then, since $W = \Delta i\sigma$,

$$h\Delta = W + w, \quad \text{therefore} \quad \frac{h}{i} = \sigma + \frac{w}{i\Delta}.$$

Let $\dfrac{h}{i} = \mu$, then $\mu = \sigma\dfrac{1+n}{n}$.

Further, let $\nu = 1-\mu$. Through the centroid draw lines as in the figure parallel to the sides of the triangle, and the line PQ so that $ABPQ = \mu\Delta$, $CPQ = \nu\Delta$. Let G be the centroid of PCQ, and draw the straight line $M_1G_1OGM_2$. If $\mu OG_1 = \nu OG$, then G_1 is the centroid of ABPQ. Now if the weight w be placed at M_1, and if

$w \,.\, M_1G_1 = OG_1 \,.\, W$, then G_1 will be the centroid of the whole mass $W + w$, and the area PABQ will begin to be immersed, its centroid being *also* G_1, so that w is just supported. But if w be placed any further from O, the lamina will upset. As the point M_1 moves from the line OF to the line OF′, the line PQ (cutting off always the same area) shifts from the position in which Q corresponds or coincides with B, and $CP = \nu b$, to the position in which P coincides with A, and $CQ = \nu a$; and the centroid G describes an arc of a hyperbola HCK terminating on OEOD.

Now, since $\dfrac{OM_1}{OG_1} = 1 + n$, $\dfrac{OG_1}{OG} = \dfrac{\nu}{\mu}$, therefore $\dfrac{OM_1}{OG} = (1+n)\dfrac{\nu}{\mu}$, it is clear that M_1 describes a similar hyperbolic curve between OF and OF′, and the area swept out by OM_1 is equal to $HOKG \times (1+n)^2 \,.\, \dfrac{\nu^2}{\mu^2}$; but when w is placed in one of the parallelogram spaces, as at M_2 in OCED, then for equilibrium the line PQ must be drawn so that $PQC = \mu\Delta$, and this is the portion immersed, $PABQ = \nu\Delta$ being then out of water. In this case, again, the centroid G of PQC must coincide with the centre of gravity of w at M_2 and W at O, so that $OM_2 \,.\, w = OG\,(W + w)$. In this case, again, G traces out an hyperbola, and the corresponding area swept out by OM_2 between OE and OD must be $= HOKG \times (1+n)^2$. In this case the HOKG is the same as before after interchanging μ and ν.

It appears then that the locus of M, or the limiting position of w, is a series of six connected hyperbolic arcs forming a closed curve. If the area of this closed curve $= \Omega$, then, since w may be placed with safety anywhere inside this area Ω, the probability that the lamina is *not* upset is $\Omega \div \Delta$. It will suffice to find the area of HOKG. Measure x and y along OE, OD; then, supposing C out of water,

$$(a - 3x)(b - 3y) = CQ \,.\, CP = \nu ab$$

is the equation of the hyperbola; also $OH = \frac{1}{3}a\mu$, $OK = \frac{1}{3}b\mu$, which give the limits of integration. Hence the area is

$$\sin C \left[\frac{bx}{3} + \frac{\nu ab}{9}\log(a - 3x)\right]_0^{\frac{1}{3}a\mu} = \frac{2\Delta}{9}(\mu + \nu\log\nu).$$

Hence the sum of the two areas swept out by OM_1 in the first case and OM_2 in the second case is, using the multiplier given above,

$$\frac{2\Delta}{9}(1+n)^2\left\{\frac{\nu^2}{\mu} + \frac{\nu^3}{\mu^2}\log\nu + \nu + \mu\log\mu\right\}$$

$$= \frac{2\Delta}{9}\frac{n^2}{\sigma^2}\left\{\nu\mu + \nu^3\log\nu + \mu^3\log\mu\right\};$$

and the chance the lamina does not upset is thus $\times 3 \div \Delta$,

or
$$p = \frac{2}{3}\frac{n^2}{\sigma^2}\left\{\nu\mu + \mu^3\log\mu + \nu^3\log\nu\right\}.$$

This, however, does not complete the solution, for the values of w and σ, &c. may be such that the closed curve is not wholly contained within the triangle. It may be easily shewn that it will touch each side of the triangle at its middle point if $2(1+n)\nu + \nu^{\frac{1}{2}} = 1$.

If the value of w be less than that which satisfies this equation, the curve will cut each side twice. If, for instanee, $\nu(1+n) = 1$, it will pass

through the six points D, D′, E, E′, F, F′; this condition may be also stated thus,
$$\sigma = \left(\frac{n}{1+n}\right)^2, \quad \text{or} \quad \frac{w}{W} = \frac{1}{\sigma^{\frac{1}{2}}} - 1,$$
which is the same as the condition that the man may go to the end of the plank; and if we express the condition that the curve traced by M_2 in the second case passes through C, we get
$$\left(\frac{n}{n+1}\right)^2 = \sigma, \quad \text{or} \quad \frac{w}{W} = \frac{1}{\sigma^{\frac{1}{2}}} - 1.$$
With this condition w may be placed anywhere on the triangle without upsetting it.

5389. (By Professor Townsend, F.R.S.)—In a trinodal quartic curve, in a plane, shew, by any method, that the four conics through the three nodes which contain the four bitangent chords touch in pairs at the three nodal points, and, by their lines of passage through them, divide harmonically at once the three nodal angles of the curve and the three angles of the nodal triangle.

Solution by the Proposer.

By inversion, in the extended sense employed by Dr. Salmon when treating of trinodal quartics in his *Higher Plane Curves* (2nd Ed., Art. 283 *et seq.*), the above becomes transformed into the well-known property that, for an arbitrary conic and triangle in its plane, the four chords of double contact, with the conic, of the four bitangent conics which pass through the three vertices of the triangle, divide harmonically at once the three sides of the triangle and the three chords determined by them in the conic; and therefore &c.

5676. (By R. Tucker, M.A.)—In the "ambiguous case" of triangles, a, b, A being given, prove that the sum of the radii of the two inscribed circles, and of the circles escribed to the side a, is equal to twice the common altitude of the triangles.

Solution by Professor Morel; H. L. Orchard, B.A., L.C.P.; *and others.*

Les formules qui donnent les rayons r et ρ des cercles inscrit et ès-inscrit touchant le côté a sont
$$r = \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A}, \quad \rho = \frac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A}.$$

En remarquant que, B et B′, C et C′ étant les angles inconnus qui correspondent aux deux solutions, on a

$$B+B'=180^\circ,\quad \text{et}\quad B+C=B'+C',$$

on en tire facilement les valeurs suivantes; où n'entrent que les angles A et B: $r=\dfrac{a\sin\frac{1}{2}B\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}A}$, $r'=\dfrac{a\cos\frac{1}{2}B\sin\frac{1}{2}(B-A)}{\cos\frac{1}{2}A}$,

$$\rho=\frac{a\cos\frac{1}{2}B\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}A},\quad \rho'=\frac{a\sin\frac{1}{2}B\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}A}.$$

En ajoutant les quatre valeurs, on trouve, toute reduction faite,

$$r+r'+\rho+\rho'=2a\sin B=2p_1.$$

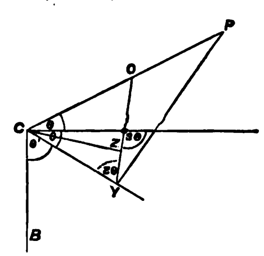
On voit en outre que $r+\rho'=r'+\rho=a\sin B=$ la hauteur commune.

5579. (By W. H. H. Hudson, M.A.)—Prove that the pedal of the evolute of the lemniscate $r^2=a^2\cos 2\theta$ is $r^2=a^2\sin^2\frac{2}{3}\theta\cos^2\frac{2}{3}\theta$.

Solution by W. Gallatly, B.A.; J. O. Jelly, M.A.; *and others.*

Consider the lemniscate as the pedal of an equilateral hyperbola with regard to the centre, that is, the locus of Y.

Bisect CP in O; OY is normal to lemniscate. Then, if CZ be drawn to OY, and $\angle ZCB=\theta'$; we have $\theta'=3\theta$.



Now $\quad CZ^2=CY^2.\sin^2 2\theta$

$\qquad\qquad = a^2\cos 2\theta.\sin 2\theta,$

therefore $\quad r'=a^2\sin^2\frac{2}{3}\theta'\cos\frac{2}{3}\theta'$,

which is the equation of locus of Z.

5631. (By Professor Lloyd Tanner, M.A.)—Show (1) that no power of 3 is of the form $13n-1$; and (2) find the lowest power of 3 which is of the form $29n-1$.

Solution by Christine Ladd; H. L. Orchard, B.A., L.C.P.; *and others.*

1. It is required to determine the value of n which satisfies the congruence $\quad 3^n\equiv -1\equiv 12 \pmod{13}$ ....(a).

The least value of n which satisfies the congruence $3^n\equiv 1 \pmod{13}$

is $n=3$. Therefore the residues of the series consisting of the powers of 3 have only the following three distinct values:—

$$3^0 \equiv 1 \text{ (mod. 13)}, \quad 3^1 \equiv 3 \text{ (mod. 13)}, \quad 3^2 \equiv 9 \text{ (mod. 13)}.$$

As no one of these residues is 12, the congruence (a) is impossible.

2. The residues corresponding to the successive indices of powers of 3 are the following:—

Ind.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Res.	3	9	27	23	11	4	12	7	21	5	15	16	19	28

From this it appears that 14 is the least value of n which satisfies the congruence $$3^n \equiv -1 \equiv 28 \text{ (mod. 29)}.$$

Otherwise, since 28 is the least value of n which satisfies the congruence

$$3^n \equiv 1 \text{ (mod. 29)},$$

and since there are in consequence 28 incongruent residues, it is sufficient to notice the first residue whose square is congruent to 28 with respect to the modulus 29. Thus,

$$[3^7 \equiv 12 \text{ (mod. 29)}]^2 \text{ gives } 3^{14} \equiv 144 \equiv 28 \text{ (mod. 29)}.$$

5643. (By W. S. B. Woolhouse, F.R.A.S.)—Any given triangle may be orthogonally projected from an equilateral triangle, or it may be orthogonally projected into an equilateral triangle; determine, by an easy geometrical construction, the magnitude of the equilateral triangle in each case.

Solution by Robert Rawson.

In the first case, I apprehend that the given triangle ABC, whose sides are a, b, c respectively, is the orthogonal projection of the equilateral triangle AB′C′ to be found. In the second case, the object is to project orthogonally the given triangle ABC into an equilateral triangle AB′C′.

First, then, let $x =$ the side of the equilateral triangle AB′C′ required. Hence BB′, CC′ are each perpendicular to the plane ABC,

therefore $$(x^2-c^2)^{\frac{1}{2}} = (x^2-b^2)^{\frac{1}{2}} + (x^2+a^2)^{\frac{1}{2}},$$

whence $$2(a^4+b^4+c^4)\,x^2-3x^4 = 4a^2b^2\sin^2 c \quad\text{.....................(1).}$$

Secondly, let $y =$ the side of the equilateral triangle AB′C′ required, then BB′, CC′ are each perpendicular to the plane AB′C′;

therefore $$(c^2-y^2)^{\frac{1}{2}} = (b^2-y^2)^{\frac{1}{2}} + (a^2-y^2)^{\frac{1}{2}},$$

whence $$2(a^2+b^2+c^2)\,y^2-3y^2 = 4a^2b^2\sin^2 c \quad\text{..................... (2).}$$

It is inferred from (1) and (2) that the values of x, y are roots of the same quadratic, that is to say, of (1).

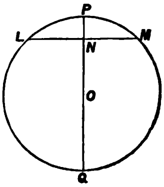
Put H = one-third of the number of square units in the sum of the areas of the equilateral triangles described upon the sides of the given triangle ABC;

Δ = number of square units in ABC;

A = number of square units in the required equilateral triangles;

then $3H = \frac{1}{4}(a^2+b^2+c^2)\sqrt{3}$; $2\Delta = ab\sin c$; and $4A = \sqrt{3}\,.\,x^2$.

Substituting these values in (1), it becomes $2H\,.\,A - A^2 = \Delta^2$............(3).



Equation (3) suggests the following construction:— With H linear units as radius, draw the circle OPMQ; insert LM equal to 2Δ linear units; bisect LM in N; join NO, and produce NO to meet the circle in P, Q; then the linear units in PN and QN are the square units, in the second and first cases, of the equilateral triangles required.

The area of the given triangle is a mean proportion between the areas of the equilateral triangles.

The sum of the areas of the equilateral triangles is equal to two-thirds the sum of the equilateral triangles described upon the sides of the given triangle.

5670. (By T. Cotterill, M.A.)—Of the nine points of intersection of two triangles inscribed in a conic, the line passing through two of these points not on the same side is a Pascal line of the system of six points on the conic.

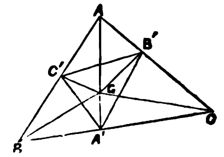
Solution by Christine Ladd.

Let the vertices of the two triangles be A, C, E and B, D, F respectively. Then the line joining $\begin{pmatrix}\mathrm{CE}\\\mathrm{BD}\end{pmatrix}$ to $\begin{pmatrix}\mathrm{FB}\\\mathrm{AC}\end{pmatrix}$ is the Pascal ACEFBD, the line joining $\begin{pmatrix}\mathrm{FB}\\\mathrm{EA}\end{pmatrix}$ to $\begin{pmatrix}\mathrm{CE}\\\mathrm{DF}\end{pmatrix}$ is the Pascal AECBFD, &c.

[Mr. Cotterill remarks that this is part of a series of properties of Pascal's hexagram of which he gave an account to the Mathematical Society, but afterwards found that a paper had been written by Veronese on the same subject. Mr. Cotterill adds that he has not yet been able to procure a copy of Veronese's paper, though the abstract shows it is very interesting, but on a totally different plan from his own.]

5562. (By D. Edwardes.)—If the cotangents of the angles of a triangle are in arithmetical progression, prove that the cotangents of the angles that the sides subtend at the centroid are also in arithmetical progression.

Solution by T. MITCHESON, B.A., L.C.P.; E. W. SYMONS; *and others.*



Let G be the centroid, GA', GB', GC' perpendiculars on the sides; then we have
$\angle BGC = BAC + GBA + GCA = BAC + B'A'C'$
(since the sets of points GB'A'C, GC'A'B, are respectively concyclic);
therefore $B'A'C' = A' - A$ (if $BGC = A'$),
$A'B'C' = B' - B$, $B'C'A' = C' - C$;

therefore $$\frac{\sin(A'-A)}{B'C'} = \frac{\sin(B'-B)}{C'A'} = \frac{\sin(C'-C)}{A'B'},$$

but $B'C' = GA \cdot \sin A$, &c.; therefore $\dfrac{\sin(A'-A)}{GA \sin A} =$ &c. ... = &c.,

or $\Delta BGC(\cot A' - \cot A) = \Delta CGA(\cot B' - \cot B) = \Delta AGB(\cot C' - \cot C)$,
therefore $\cot A' - \cot A = \cot B' - \cot B = \cot C' - \cot C$
$= \frac{1}{2}(\cot A' + \cot C') - \frac{1}{2}(\cot A + \cot C) = \frac{1}{2}(\cot A' + \cot C') - \cot B$,
therefore $\cot B' = \frac{1}{2}(\cot A' + \cot C')$, which proves the theorem.

5406. (By J. J. WALKER, M.A.)—If O_1, O_2, O_3 are the centres of circles escribed to the spherical triangle ABC; prove that

$$\frac{\cos O_2O_1O_3}{\sin\frac{1}{2}A} = \frac{\cos O_3O_2O_1}{\sin\frac{1}{2}B} = \frac{\cos O_1O_3O_2}{\sin\frac{1}{2}C} = \frac{1-\cos A-\cos B-\cos C}{4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}.$$

Solution by R. TUCKER, M.A.; Prof COCHEZ; *and others.*

Let $\angle O_2O_1O_3 = \omega_1$; then we have

$$\cos\omega + \sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \cos a\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$
$$\cos A + \cos B\cos C = \cos a\sin B\sin C.$$

Eliminating $\cos a$, we get

$$4\cos\omega_1\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \cos A + \cos B\cos C - (1-\cos B)(1-\cos C)$$
$$= -(1-\cos A-\cos B-\cos C);$$

therefore $$\frac{\cos\omega_1}{\sin\frac{1}{2}A} = -\frac{1-\cos A-\cos B-\cos C}{4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}.$$

The remaining equations follow from symmetry.

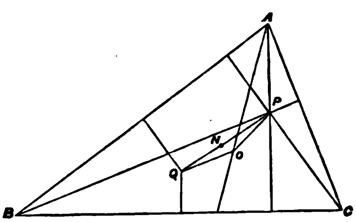
5398. (By the EDITOR.)—If, in a plane triangle, O be the centre and r the radius of the inscribed circle, P the orthocentre, and ρ the radius

of the circle inscribed in the orthocentric triangle, and Q the centre and R the radius of the circumscribed circle,—so that OPQ may be conveniently called the *triangle of centres*,—find expressions for the sides and area of the triangle of centres, and prove that

$$OP \cdot OQ \cos POQ = R\rho - Rr + r^2.$$

Solution by R. F. DAVIS, B.A.; CHRISTINE LADD; *and others.*

The nine-point circle of the triangle ABC has its centre at the middle point (N) of PQ, and touches the inscribed circle, so that $ON = \frac{1}{2}R - r$. Now we have



$$OQ^2 = R^2 - 2Rr,$$

a fundamental formula; while the analogue for the orthocentric triangle is

$$PN^2 = (\tfrac{1}{2}R)^2 - 2(\tfrac{1}{2}R)\rho,$$

and $PQ^2 = 4PN^2 = R^2 - 4R\rho$.

But $OP^2 + OQ^2 = 2ON^2 + 2PN^2$,

hence we find $\quad OP^2 = 2(r^2 - R\rho).$

Now $\quad OP \cdot OQ \cos POQ = \frac{1}{2}(OP^2 + OQ^2 - PQ^2) = R\rho - Rr + r^2.$

The area may be found either by calculating $\frac{1}{2}OP \cdot OQ \sin POQ$, or by finding the perpendicular from O on PQ. One form of the result is

$$4\Delta^2 = (R^2 - 2Rr)(r^2 - R\rho) - R^{-1}\rho\,(R^2 - 2Rr)^2 - (r^2 - R\rho)^2,$$

which vanishes of course when $R = \frac{1}{2}r = \frac{1}{4}\rho$.

When multiplied out, the result becomes

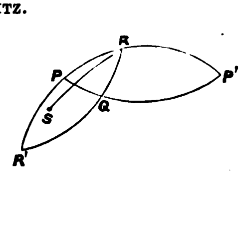
$$4\Delta^2 = R^2r^2 + 6R^2r\rho - R^2\rho^2 - 2R^3\rho - 2Rr^3 - 2Rr^2\rho - r^4.$$

[See *Reprint*, Vol. XXVII., pp. 85—87.]

5668. (By ELIZABETH BLACKWOOD.)—Four points P, Q, R, S being taken at random in the surface of a given sphere, show that the chance that the arcs PQ and RS (that is, the smaller arcs of the great circles through those points) will intersect, is $\frac{1}{8}$.

I. *Solution by* E. B. SEITZ.

Produce the arcs PQ and PR to P′, and RP and RQ to R′.



Now, if S be taken anywhere in the triangle PQR′, the arcs PQ and RS will intersect.

Let $PQ = x$, $PR = y$, $\angle QPR = \theta$, and let the radius of the sphere be unity. Then area of lune $PP' = 2\theta$; an element of the surface at Q is $2\pi \sin x\,dx$, and at R it is $d\theta \sin y\,dy$; the limits for each of the variables are 0 and π.

The average of the area of the triangle PQR is equal to that of the triangle PQR, which is equal to one-half of the average of the area of the lune PP′. Hence the required chance is

$$p = \frac{\int_0^\pi \int_0^\pi \int_0^\pi \theta\, d\theta \,.\, 2\pi \sin x\, dx \sin y\, dy}{\int_0^\pi \int_0^\pi \int_0^\pi 4\pi\, d\theta \,.\, 2\pi \sin x\, dx \sin y\, dy}$$

$$= \frac{1}{16\pi^2} \int_0^\pi \int_0^\pi \int_0^\pi \theta\, d\theta \sin x\, dx \sin y\, dy$$

$$= \frac{1}{8\pi^2} \int_0^\pi \int_0^\pi \theta\, d\theta \sin x\, dx = \frac{1}{4\pi^2} \int_0^\pi \theta\, d\theta = \frac{1}{8}.$$

II. *Solution by the* PROPOSER.

This problem may be solved without the aid of the Integral Calculus as follows:—

Let X, Y be the points at which the great circles through PQ and RS intersect. Then, always measuring in the directions PQ and RS, and taking the diameters in which the arcs PQ and RS are less than a semicircle, we shall in every case, *within the limits of a semi-circle*, pass either the point X or the point Y; so that in the one great circle we must have one or other of the four permutations (all equally probable) PXQ, PYQ, PQX, PQY, and in the other great circle we must have one or other of the four permutations (all equally probable) RXS, RYS, RSX, RSY. Again, any one of the first four permutations may be combined with any one of the second four, and out of these 16 possible combinations (all equally probable) two only are favourable, namely, the combination of PXQ with RXS and the combination of PYQ with RYS. The required chance is therefore $\frac{2}{16}$ or $\frac{1}{8}$.

[This agrees with the result which Mr. SEITZ has, in the foregoing solution, obtained with the aid of the Integral Calculus.]

5696. (By Professor MONCOURT.)—Un losange ABCO articulé a un sommet fixe O, le sommet B décrit une ligne droite; démontrer que tout point pris sur le côté BC, ou sur son prolongement, décrit une ellipse.

Solution by J. J. WALKER, M.A.

I find that the locus of a fixed point (P) on the side BC proves to be a quartic curve, which does not appear to break up. It may be easily deduced in a Quaternion form. Let α be a unit vector parallel to the line on which B moves, β the perpendicular vector to that line, from O; while

ρ, γ, δ are the vectors OP, OC, OB, and $\delta = x\alpha + \beta$. Then, if

$$\mathrm{CP} : \mathrm{PB} = m : 1-m,$$

$$\rho + (m-1)\gamma - m\delta = 0 \text{ and } 2S\gamma\delta + (T\delta)^2 = 0 \text{............}(1, 2),$$

as the condition of the equality of OC, CB.

From (1) we have $S\rho\delta + (m-1)\,S\gamma\delta + m\,(T\delta)^2 = 0$ .... (3),

and, eliminating $S\gamma\delta$ from (2), (3), we obtain

$$2S\rho\delta + (1+m)(T\delta)^2 = 0, \text{ or } 2S.\rho(x'\alpha+\beta) + (1+m)\left\{T(x'\alpha+\beta)\right\}^2 = 0 \text{...}(4),$$

in which x' is a variable scalar. To express (4) in (xy) coordinates of P, α, β being axes, let $\mathrm{OC} = a$, and b be the length of β.

Then $\quad x^2 + y^2 - m(x'^2 - b^2) - (1-m)^2 a^2 = 0,$

with $\quad (x'-x)^2 + (y-b)^2 - (1-m)^2 a^2 = 0.$

Eliminating x' from these equations, we have

$$\left\{x^2+y^2-(1-m)^2a^2\right\}\left[(1-m)^2x^2 + \left\{(1+m)\,y - 2mb\right\}^2 - (1-m^2)^2a^2\right] + 4m^2(1-m)^2a^2b^2 = 0 \text{.........}(5).$$

If $x = 0$, $\quad \left\{(1+m)\,y^2 - 2mby - (1+m)(1-m)^2a^2\right\}^2 = 0,$

i.e., β is a bitangent; if $y = 0$,

$$(1-m)^2x^4 - 2\left\{(1-m)^2(1+m^2)\,a^2 + 2m^2b^2\right\}x^2 + (1+m)^2(1-m)^4a^4 = 0;$$

results which may readily be verified from the figure.

Equation (5) may be further verified by showing that it is satisfied by the (xy) of P in the case in which P lies on OB, viz.,

$$x = \tfrac{1}{2}(1+m)(4a^2-b^2)^{\frac{1}{2}}, \quad y = \tfrac{1}{2}(1+m)\,b.$$

5662. (By Professor TOWNSEND, F.R.S.)—

If $\quad S_1 = a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy = 0,$

and $\quad S_2 = a_2x^2 + b_2y^2 + c_2z^2 + 2f_2yz + 2g_2zx + 2h_2xy = 0,$

be the equations of the asymptotic cones of two quadric surfaces having a common centre; show that their common triad of conjugate diametral planes is given by the equation,

$$\begin{vmatrix} \frac{dS_1}{dx} & \frac{dS_1}{dy} & \frac{dS_1}{dz} \\ \frac{dS_2}{dx} & \frac{dS_2}{dy} & \frac{dS_2}{dz} \\ \frac{dS_3}{dx} & \frac{dS_3}{dy} & \frac{dS_3}{dz} \end{vmatrix} = 0;$$

where $S_3 = A_1\left(\frac{dS_2}{dx}\right)^2 + \text{\&c.},$

or $\quad = A_2\left(\frac{dS_1}{dx}\right)^2 + \text{\&c.},$

A_1, B_1, &c. and A_2, B_2, &c. being the reciprocal coefficients to a_1, b_1, &c. and to a_2, b_2, &c.

I. *Solution by* W. J. C. SHARP, B.A.; Prof. EVANS, M.A.; *and others.*

The cone S_3 is the locus of the poles of the tangent planes to S_1 with respect to S_2, if the upper value be taken, or of the poles of the tangent planes to S_2 with respect to S_1, if the lower value, either of which, as appears from SALMON'S *Geometry of Three Dimensions*, p. 159, has the same system

of common conjugate planes as the two given cones; hence the diametral planes conjugate to the diameter $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$ in the three cones are

$$\lambda \frac{dS_1}{dx} + \mu \frac{dS_1}{dy} + \nu \frac{dS_1}{z} = 0, \quad \lambda \frac{dS_2}{dx} + \mu \frac{dS_2}{dy} + \nu \frac{dS_2}{dz} = 0,$$

$$\lambda \frac{dS_3}{dx} + \mu \frac{dS_3}{dy} + \nu \frac{dS_3}{dz} = 0,$$

which, as the planes are identical, lead to the equation in the question, and this, being an homogeneous equation of the third degree, represents three planes, *i.e.*, the three mutually conjugate planes required.

II. *Solution by the* PROPOSER.

The above being the well-known equation of the Jacobian of the three surfaces S_1, S_2, S_3, represents consequently, when they have such, their common triad of conjugate diametral planes; but S_3 representing, as is also well known, the reciprocal of S_1 with respect to S_2 in its first form, and that of S_2 with respect to S_1 in its second form, the three surfaces S_1, S_2, S_3 have consequently, in either case, a common triad of conjugate diametral planes; and therefore, &c.

N.B.—If either S_1 or $S_2 = x^2 + y^2 + z^2$, and represent in consequence a point sphere, the above equation becomes, as it manifestly ought, that given by Dr SALMON, in his *Geometry of Three Dimensions*, for the triad of principal diametral planes of the other.

5700. (By Colonel CLARKE, C.B., F.R.S.)—Required an investigation of the following problem:—You can ascertain whether a boiled egg is hard-boiled or soft by this process,—Lay the egg on a smooth horizontal surface; spin it (initially round a vertical axis perpendicular to its axis of figure); then, if soft it continues to rotate in that way (Fig. 1); but if hard it soon gets up on end thus (as shown in Fig. 2):—

FIG. 1.

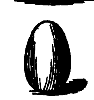


FIG. 2.

[Col. CLARKE remarks that any mathematician can try the experiment at breakfast, adding the necessary proviso that it is well not to break the egg.]

Solution by H. L. ORCHARD, B.A., L.C.P.

If, when the egg is spun as in the question, the contents are *liquid*, there is no vertical force produced upon its centre of gravity,—the interior of the shell being supposed *smooth*. But, if the contents are *solid*, we shall have to consider a resultant force made up of the (horizontal) spinning force and of the moment of inertia due to rotation about the centre of gravity; this resultant will (by the parallelogram of forces) tend to tilt up the centre of gravity (situated towards the broad end) and therefore to raise the egg up on that end.

5488. (By C. K. PILLAI.)—An ellipse is placed with its major axis vertical; find the radius vector by which a particle will descend in the shortest time from the upper focus to the curve.

Solution by H. POLLEXFEN, B.A.; C. BICKERDIKE; *and others.*

Let S be the upper focus. From S draw a chord SP to the ellipse equal in length to the latus rectum. Let PG be the normal at P, SG being the major axis. From G draw GL perpendicular to SP; then PL = the semi-latus rectum, therefore SP is bisected in L; whence PG = SG, and a circle, centre G and radius GS, has its highest point and touches the ellipse at P; therefore SP is the chord required.

5371. (By S. ROBERTS, M.A.)—Find the equation of the curve along which the faisceaux of curves $U+\alpha V=0$, $S+\beta T=0$ touch, α, β being parameters.

Solution by the PROPOSER; H. T. GERRANS, B.A.; *and others.*

We have the conditions,

$$\left(\frac{du}{dx}+\alpha\frac{dv}{dx}\right)\left(\frac{dS}{dy}+\beta\frac{dT}{dy}\right)-\left(\frac{du}{y}+\alpha\frac{dv}{dy}\right)\left(\frac{dS}{dx}+\beta\frac{dT}{dx}\right)=0,$$

$$\left(\frac{du}{dx}+\alpha\frac{dv}{dx}\right)\left(\frac{dS}{dz}+\beta\frac{dT}{dz}\right)-\left(\frac{du}{dz}+\alpha\frac{dv}{dz}\right)\left(\frac{dS}{dx}+\beta\frac{dT}{dx}\right)=0,$$

$$U+\alpha V=0.$$

Eliminating β and removing the factor $\frac{du}{dx}+\alpha\frac{dv}{dx}$, and then eliminating α, we get

$$\left(V\frac{du}{dx}-U\frac{dv}{dx}\right)\left(\frac{dS}{dy}\frac{dT}{dz}-\frac{dT}{dy}\frac{dS}{dz}\right)+\left(V\frac{du}{dy}-U\frac{dv}{dy}\right)\left(\frac{dT}{dx}\frac{dS}{dz}-\frac{dS}{dx}\frac{dT}{dz}\right)$$
$$+\left(V\frac{du}{dz}-U\frac{dv}{dz}\right)\left(\frac{dT}{dy}\frac{dS}{dx}-\frac{dS}{dy}\frac{dT}{dx}\right)=0.$$

5686. (By W. H. H. HUDSON, M.A.)—If $y=\cos^n x$, prove that

$$\left(\frac{d^2}{dx^2}+n^2\right)\left\{\frac{d^2}{dx^2}+(n-2)^2\right\}\left\{\frac{d^2}{dx^2}+(n-4)^2\right\}\ldots\ldots\left(\frac{d^2}{dx^2}+2^2\right)y=n!$$

when n is even, and write down the corresponding theorem if n be odd.

Solution by J. HAMMOND, M.A.; Prof. JOHNSON, M.A.; *and others*.

If $y = \cos^n x$, we have $\dfrac{dy}{dx} = -n\cos^{n-1}x \,.\, \sin x$,

$$\frac{d^2y}{dx^2} = n(n-1)\cos^{n-2}x \,.\, \sin^2 x - n\cos^n x = n(n-1)\cos^{n-2}x - n^2\cos^n x,$$

therefore $$\left(\frac{d^2}{dx^2} + n^2\right) y = n(n-1)\cos^{n-2}x.$$

Hence, when n is even,

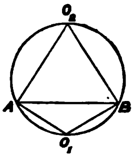
$$\left(\frac{d^2}{dx^2} + n^2\right)\left\{\frac{d^2}{dx^2} + (n-2)^2\right\}\left\{\frac{d^2}{dx^2} + (n-4)^2\right\} \dots\dots \left(\frac{d^2}{dx^2} + 2^2\right) y = n\,!,$$

and when n is odd,

$$\left(\frac{d^2}{dx^2} + n^2\right)\left\{\frac{d^2}{dx^2} + (n-2)^2\right\} \dots\dots \left(\frac{d^2}{dx^2} + 3^2\right) y = n\,!\cos x.$$

5551. (By Professor MONCK, M.A.)—A chord of given length is drawn in a given circle; find the chance that a second chord, drawn at random, will intersect the first chord within the circle.

Solution by the PROPOSER.



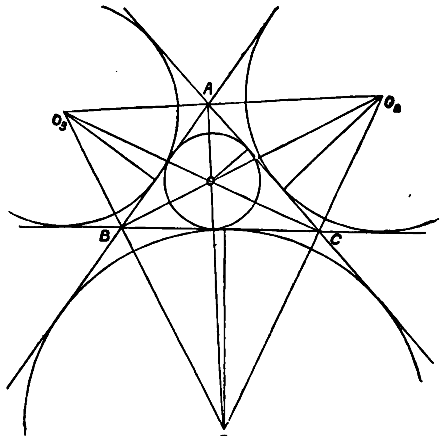
Let AB be the given chord, and suppose random chords to be drawn through every point on the circumference. The chance that such a point will lie in the arc AO_2B is evidently as the arc AO_2B to the whole circumference. Let O_2 be such a point. Join AO_2, BO_2, and then evidently, if the random chord falls within the angle AO_2B (which is the same for any point in this arc), it will intersect AB; otherwise not. The chance of intersection is therefore as the angle AO_2B to two right angles, or as the arc AO_1B to the circumference, and therefore the chance of the random point lying in the arc AO_2B *and* a random chord through it intersecting AB is as the product of the arcs AO_2B, AO_1B to the square of the circumference. Applying a similar process to the arc AO_1B, we find the total chance of intersection is $\dfrac{2 \text{ arc } AO_1B \times \text{arc } AO_2B}{(\text{Circumference})^2}$.

5560. (By S. CONSTABLE.)—If O, O_1, O_2, O_3 are the centres of the inscribed and escribed circles of a triangle ABC; prove that the areas

$$O_1O_2O_3 = 2Rs, \quad OO_2O_3 = 2Rs_1, \text{ \&c., where } s_1 = s - a, \text{ \&c.}$$

Solution by the EDITOR.

The triangle ABC is the orthocentric triangle of the triangle $O_1O_2O_3$, and the radius of the circle that circumscribes the triangle $O_1O_2O_3$ is 2R;



hence, by what is proved in BOOTH's *Geometrical Methods*, Vol. II., p. 313, eq. (*e*), the area of this triangle is $2Rs$.

Again, $$\Delta O_2OO_3 = \tfrac{1}{2} OA \,.\, O_2O_3 = \tfrac{1}{2} r \operatorname{cosec} \tfrac{1}{2}A \,.\, 4R \cos \tfrac{1}{2}A$$
$$= 2Rr \cot \tfrac{1}{2}A = 2Rr \frac{s_1}{r} = 2Rs_1.$$

Similarly, $\Delta O_3OO_1 = 2Rs_2$, and $\Delta O_1OO_2 = 2Rs_3$; hence, by addition, we obtain $\Delta O_1O_2O_3 = 2R(s_1+s_2+s_3) = 2Rs$, which verifies the first part of the theorem.

5692. (By Professor WOLSTENHOLME, M.A.)—

If $\quad U_n \equiv \int_0^{\frac{1}{2}\pi} (\log \tan x)^{2n}\, dx,$ prove that

$$U_1 = (\tfrac{1}{2}\pi)^3, \qquad U_2 = 5\,(\tfrac{1}{2}\pi)^5, \qquad U_3 = 61\,(\tfrac{1}{2}\pi)^7,$$
$$U_4 = 1385\,(\tfrac{1}{2}\pi)^9, \qquad U_5 = 50521\,(\tfrac{1}{2}\pi)^{11}, \qquad U_6 = 2702765\,(\tfrac{1}{2}\pi)^{1}$$
$$U_7 = 199360981\,(\tfrac{1}{2}\pi)^{15}, \qquad U_8 = 19391512145\,(\tfrac{1}{2}\pi)^{17}, \quad \&c.$$

and generally, $U_n =$ an odd number $\times (\frac{1}{2}\pi)^{2n+1}$, the last digit in the odd number being alternately 1 and 5.

I. *Solution by* E. B. ELLIOTT, M.A.; Professor ARMENANTE; *and others.*

Using the transformation $\log \tan x = y$, we have

$$U_n = \int_{-\infty}^{\infty} y^{2n} \cdot \frac{\epsilon^y\, dy}{1+\epsilon^{2y}} = \int_{-\infty}^{\infty} z^{2n} \cdot \frac{\epsilon^{-z}\, dz}{1+\epsilon^{-2z}}, \text{ replacing } y \text{ by } -z,$$

$$= \int_{-\infty}^{\infty} z^{2n}\, \epsilon^{-z} \left\{1-\epsilon^{-2z}+\epsilon^{-4z}-\epsilon^{-6z}+\dots\right\} dz.$$

Now $$\int_{-\infty}^{\infty} z^{2n}\, \epsilon^{-az}\, dz = \frac{2}{a^{2n+1}}\, \Gamma(2n+1).$$

Therefore $$U_n = 2\Gamma(2n+1)\left\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+}}+\dots\right\}.$$

But (Herschel's *Finite Differences*, p. 92) the series in the bracket equals $(\frac{1}{2}\pi)^{2n+1} \times \frac{1}{2}$, the coefficient of θ^{2n} in the expansion of $\sec\theta$. Hence, by developing to a few terms, the values of U_n for small values of n given in the question are found. For the general expression in terms of Bernoulli's numbers, see the part of Herschel referred to.

[Prof. Wolstenholme states that the value of U_1 is easily found as follows:— $\log(2\cos x) = (\cos 2x - \frac{1}{2}\cos 4x + \frac{1}{3}\cos 6x - \dots)$,

$\log(2\sin x) = -(\cos 2x + \frac{1}{2}\cos 4x + \frac{1}{3}\cos 6x + \dots)$,

if x lie between 0 and $\frac{1}{2}\pi$; hence we have

$$\log\tan x = -2\,(\cos 2x + \tfrac{1}{3}\cos 6x + \dots),$$

$$\int_0^{\frac{1}{2}\pi} (\log\tan x)^2\, dx = \int_0^{\frac{1}{2}\pi} 4\left(\cos^2 2x + \frac{1}{3^2}\cos^2 6x + \frac{1}{5^2}\cos^2 10x + \dots\right) dx$$

$$= 2\,.\,\tfrac{1}{2}\pi\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right) = \frac{\pi^3}{8},$$

but the others would not easily come from this; and he adds that

$$U_n = \int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta}^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta}^{\frac{1}{2}\pi}\int_0^{\theta} \dots\dots \tfrac{1}{2}\pi\, (d\theta)^{2r},$$

so that, he remarks, the value of U_n can always be found, *with patience.*]

II. *Solution by the* Proposer; H. Stabenow; *and others.*

We have

$$\int_0^{\infty} \frac{x^m + x^{-m}}{1+x^2}\, dx = 2\int_0^{\infty} \frac{x^m}{1+x^2}\, dx = \frac{2}{m+1}\int_0^{\infty} \frac{dx}{1+x^{(m+1)^{-2}}}$$

$$= \frac{2}{m+1}\, \frac{\pi}{(m+1)^{-2}\sin\frac{1}{2}(m+1)\pi}, \text{ if } m<1, = \frac{\pi}{\cos\frac{1}{2}(m\pi)}.$$

Also $$\int_0^{\infty} \frac{x^m + x^{-m}}{1+x^2}\, dx = 2\int_0^{\infty} \left\{1+\tfrac{1}{2}m^2(\log x)^2 + \frac{m^4}{4\,!}(\log x)^4 + \dots\right\} \frac{dx}{1+x^2}$$

$$= 2\int_0^{\frac{1}{2}\pi} \left\{1+\tfrac{1}{2}m^2(\log\tan x)^2 + \frac{m^4}{4\,!}(\log\tan x)^4 + \dots\right\} dx$$

$$= 2\left\{U_0 + \frac{m^2}{2\,!}U_1 + \dots + \frac{m^{2n}}{2n\,!}U_n + \dots\right\}.$$

Hence $\frac{U_n}{2n!} = \frac{1}{2}\pi \times$ coefficient of m^{2n} in the expansion of $\sec \frac{1}{2}m\pi$

$= (\frac{1}{2}\pi)^{2n+1} \times$ coefficient of m^{2n} in the expansion of $\sec m$,

or $U_{2n} = (\frac{1}{2}\pi)^{2n+1} \times$ coefficient of $\frac{m^{2n}}{2n!}$ in the expansion of $\sec m$

$$= (\tfrac{1}{2}\pi)^{2n+1} \left.\frac{d^{2n}(\sec m)}{dm^{2n}}\right]_{m=0}.$$

Now $$\frac{d^2(\sec m)}{dm^2} = 2\sec^3 m - \sec m;$$

$$\frac{d^4}{dm^4}(\sec m) = 24\sec^5 m - 20\sec^3 m + \sec m; \text{ \&c.};$$

whence it easily follows that in general all the coefficients of the different powers of $\sec m$ are even, except that of $\sec m$ which is ± 1, so that

$$\frac{d^{2n}}{dm^{2n}}(\sec m)_{m=0}$$

is always an odd number.

Mr. R. Rawson first pointed out to me that

$$U_n = 2n! \times (\tfrac{1}{2}\pi)^{2n+1} \times \text{coefficient of } \theta^{2n}$$

in the expansion of $\sec\theta$, and corrected my results from Herschel's tables.

In general, $\frac{1}{4}\pi = \cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots$, if $\cos\theta$ be positive,

whence $$\tfrac{1}{4}\pi\theta = \sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \ldots,$$

$$\tfrac{1}{8}\pi(\tfrac{1}{4}\pi^2 - \theta^2) = \cos\theta - \frac{1}{3^3}\cos 3\theta - \frac{1}{5^3}\cos 5\theta - \ldots,$$

$$\text{\&c.} = \text{\&c.}$$

Now, $$U_n = \int_0^\infty \frac{(\log x)^{2n}}{1+x^2}dx = 2\int_0^1 \frac{(\log x)^{2n}}{1+x^2}dx$$

$$= 2\int_0^1 (1 - x^2 + x^4 - \ldots)(\log x)^{2n} dx$$

$$= 2 \,.\, 2n! \left\{\frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots\right\}$$

$$= \tfrac{1}{2}\pi \,.\, 2n! \int_0^{\frac{1}{2}} \int_0^\theta \int_\theta^{\frac{1}{2}\pi} \int_0^\theta \ldots \int_\theta^{\frac{1}{2}\pi} \int_0^\theta d\theta^{2r}.$$

III. *Solution by* J. Hammond, M.A.; Prof. Moncourt; *and others.*

Putting $\log\tan x = \frac{1}{2}\pi\theta$, we have $\frac{\sec^2 x\,dx}{\tan x} = \frac{1}{2}\pi d\theta$, or $dx = \frac{\frac{1}{2}\pi\, d\theta}{e^{\frac{1}{2}\pi\theta} + e^{-\frac{1}{2}\pi\theta}}$.

Thus $$\int_0^{\frac{1}{2}\pi} (\log\tan x)^{2n} dx = (\tfrac{1}{2}\pi)^{2n+1} \int_0^\infty \frac{\theta^{2n}\,d\theta}{e^{\frac{1}{2}\pi\theta} + e^{-\frac{1}{2}\pi\theta}}.$$

The value of this integral (See BOOLE's *Finite Differences*, 2nd Edition, p. 110) is $\frac{E_{2n}}{2}$, where E_{2n} is the coefficient of $\frac{x^{2n}}{2n!}$ in the expansion of sec x.

Thus $$\int_0^{\frac{1}{2}\pi} (\log \tan x)^{2n}\, dx = \tfrac{1}{2}E_2 \; (\tfrac{1}{2}\pi)^{2n+1}.$$

In the *Proceedings* of the London Mathematical Society, Vol. VII., p. 13, I have shown that

$$E_{2n} = \begin{vmatrix} 1, & 1, & 0, & 0, & \ldots \\ 1, & 6, & 1, & 0, & \ldots \\ 1, & 15, & 15, & 1, & \ldots \\ 1, & 28, & 70, & 28, & \ldots \\ & \ldots & \ldots & \ldots & \end{vmatrix} \; (n \text{ rows}) \ldots\ldots\ldots\ldots\ldots\ldots \; (1),$$

the determinant (1) being obtained from the determinant of the binomial coefficients

$$\begin{vmatrix} 1, & 2, & 1, & 0, & 0, & 0, & 0, & 0, & \ldots \\ 1, & 3, & 3, & 1, & 0, & 0, & 0, & 0, & \ldots \\ 1, & 4, & 6, & 4, & 1, & 0, & 0, & 0, & \ldots \\ 1, & 5, & 10, & 10, & 5, & 1, & 0, & 0, & \ldots \\ 1, & 6, & 15, & 20, & 15, & 6, & 1, & 0, & \ldots \\ 1, & 7, & 21, & 35, & 35, & 21, & 7, & 1, & \ldots \\ 1, & 8, & 28, & 56, & 70, & 56, & 28, & 8, & \ldots \\ \ldots & & \ldots & & \ldots & \ldots & \ldots & & \ldots \end{vmatrix} \; (2n \text{ rows})$$

by striking out all even rows and all even columns.

By means of (1) the successive values of E_{2n} are easily calculated. We have

$$\begin{aligned} E_2 &= 1, \\ E_4 &= 6E_2 - 1 = 5, \\ E_6 &= 15E_4 - 15E_2 + 1 = 61, \\ E_8 &= 28E_6 - 70E_4 + 28E_2 - 1 = 1385, \\ E_{10} &= 45E_8 - 210E_6 + 210E_4 - 45E_2 + 1 = 50521, \\ E_{12} &= 66E_{10} - 495E_8 + 924E_6 - 495E_4 + 66E_2 - 1 = 2702765. \end{aligned}$$

To prove that E_{2n} is an odd number, observe that the coefficients of each of the above equations occur in pairs, except when n is even, when there is a coefficient $\frac{2n!}{n!\,n!}$ occurring singly.

This coefficient becomes (writing $2m$ for n)

$$\frac{(2m+1)\,(2m+2)\ldots 4m}{1\,.\,2\,.\,3\ldots 2m} = 2\,\frac{(2m+1)\,(2m+2)\ldots(4m-1)}{1\,.\,2\ldots 2m-1} = \text{even number.}$$

Thus, if all the E's below E_{2n} are odd,

$$E_{2n} = \text{even number} \pm 1 = \text{odd}.$$

The theorem is thus completely proved.

5446. (By S. ROBERTS, M.A.)—Shew that the triangular numbers which are also squares are given by

$$\left\{\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}\right\}^2.$$

I. *Solution by* C. LEUDESDORF, M.A.

The general form of a triangular number being $\frac{1}{2}n(n+1)$, we have

$\frac{1}{2}n(n+1) = z^2$; therefore $n^2+n-2z^2 = 0$, or $(2n+1)^2-8z^2 = 1$.

Now one solution of this equation is clearly $2n+1 = 3$, $z=1$; the general value of z is therefore

$$\frac{1}{2\sqrt{8}}\left\{(3+\sqrt{8})^m-(3-\sqrt{8})^m\right\} = \frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}},$$

and the number itself, $=z^2$, is the square of this expression.

II. *Solution by the* PROPOSER; Prof. EVANS, M.A.; *and others*.

If a triangular number $\frac{1}{2}p\,(p+1)$ is also a square, we must have

(1) $p = 2P^2$, $p+1 = Q^2$, or (2) $p+1 = 2P'^2$, $p = Q'^2$.

1. We have to solve in whole numbers $Q^2-2P^2 = 1$; viz., we have

$$P = \frac{(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}}{2\sqrt{2}},\quad Q = \frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}}{2}.$$

2. We have to solve $Q'^2-2P'^2 = -1$, so that

$$P' = \frac{(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}}{2\sqrt{2}},\quad Q' = \frac{(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}}{2}.$$

Hence $\frac{1}{2}p\,(p+1)$ has the form given.

By decomposing the expression we can obtain remarkable laws of derivation of the successive squares. Thus we have the series of numbers

$$\begin{array}{cccc} 1^2.1^2, & (1+1)^2(2.1+1)^2, & (2.2+3)^2(2+3)^2, & (7+5)^2(2.5+7)^2, \\ 1, & 2^2.3^2, & 7^2.5^2, & 12^2.17^2, \end{array}$$

$$\left.\begin{array}{c}(2.12+17)^2(12+17)^2 \\ 41^2.29^2\end{array}\right\}, \text{ \&c. \&c.}$$

5630. (By Professor MONCK, M.A.)—A square number is equal to the sum of any given number of squares; show how to obtain a series of integral square numbers possessing the same property.

Solution by H. L. ORCHARD, B.A., L.C.P.; *the* PROPOSER; *and others.*

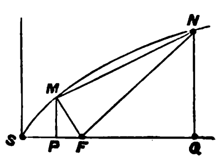
Let $a^2+b^2+c^2+\ldots = A^2$, and $a+b+c+\ldots = s$; then,
if we take $2s+2A-(n-1)\,a = a_1$, $2s+2A-(n-1)\,b = b_1$, &c.,
and $$2s+(n+1)\,A = A_1,$$
we shall have $$a_1^2+b_1^2+c_1^2+\ldots = A_1^2,$$
and we can continue the series as long as we please.

Squaring and adding, we obtain
$$a_1^2+b_1^2+c_1^2+\ldots = 4ns^2+8nsA+4nA^2-4(n-1)\,s\,(a+b+c+\ldots) \\ -4\,(n-1)\,A\,(a+b+c+\ldots)+(n-1)^2A^2,$$
or, since $a+b+c+\ldots = s$,
$$= \{4n-4(n-1)\}\,s^2+\{8n-4(n-1)\}sA+\{4n+(n-1)^2\}A^2$$
$$= 4s^2+4(n+1)\,sA+(n+1)^2A^2 = \{2s+(n+1)A\}^2 = A_1^2.$$

In forming this series, of course any of the terms may be negative as well as positive.

5364. (By the EDITOR.)—If ρ_1, ρ_2 be the focal vectors FM, FN of two points M, N on a parabola whose parameter is $4a$, δ the chord of the arc MN, and Σ the area of the parabolic sector FMN; prove that
$$\Sigma = \tfrac{1}{12}(2a)^{\frac{1}{2}}\left\{(\rho_1+\rho_2+\delta)^{\frac{3}{2}} - (\rho_1+\rho_2-\delta)^{\frac{3}{2}}\right\}.$$

Solution by Professor COCHEZ.



L'aire cherchie Σ a pour expression
$$\Sigma = \text{SNQ}-\text{SMP}-\text{PMF}-\text{QNF} \ \ldots\ldots(1).$$
Si (x_1, y_1) et (x_2, y_2) sont les coordonnées des points M et N, on a
$$y_1^2 = 4ax_1, \quad y_2^2 = 4ax_2 \ \ldots\ldots\ldots\ldots(2);$$
or $\text{SNQ} = \frac{2}{3}x_2y_2$, $\text{SMP} = \frac{2}{3}x_1y_1$;
$\text{PMF} = \frac{1}{2}(a-x_1)\,y_1$, $\text{QNF} = \frac{1}{2}(x_2-a)\,y_2$.

Portant ces valeurs dans (1), il vient, après réductions,
$$\Sigma = \tfrac{1}{6}(x_2y_2-x_1y_1) + \tfrac{1}{2}a\,(y_2-y_1).$$
Les relations (2) donnent $y_1 = 2\,(ax_1)^{\frac{1}{2}}$, $y_2 = 2\,(ax_2)^{\frac{1}{2}}$,
dès lors $$\Sigma = \tfrac{1}{3}a^{\frac{1}{2}}\,(x_2^{\frac{1}{2}}-x_1^{\frac{1}{2}})\,[x_2+x_1+(x_1x_2)^{\frac{1}{2}}+3a];$$
d'un autre côté $\delta^2 = (y_2-y_1)^2+(x_2-x_1)^2$, $\rho_1 = a+x_1$, $\rho_2 = a+x_2$,
d'où $$\rho_1+\rho_2 = 2a+x_1+x_2;$$
dès lors $$\delta^2 = 4a\,(x_2^{\frac{1}{2}}-x_1^{\frac{1}{2}})^2+({}_2x-x_1)^2.$$

Si l'on pose $\qquad \rho_1+\rho_2+\delta = 2p_1, \quad \rho_1+\rho_2-\delta = 2p_2,$

il vient $\rho_1+\rho_2 = p_1+p_2$, ou $x_1+x_2 = p_1+p_2-2a, \quad x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} = p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}-a$;

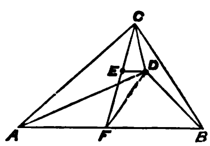
de là on tire $\qquad (x_2^{\frac{1}{2}}-x_1^{\frac{1}{2}})^2 = p_1+p_2-2p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} = (p_1^{\frac{1}{2}}-p_2^{\frac{1}{2}})^2$

alors $\quad \Sigma = \frac{1}{3}a^{\frac{1}{2}}(p_1^{\frac{1}{2}}-p_2^{\frac{1}{2}})\left[(p_1^{\frac{1}{2}}-p_2^{\frac{1}{2}})^2+3p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}\right] = \frac{1}{3}a^{\frac{1}{2}}(p_1^{\frac{3}{2}}-p_2^{\frac{3}{2}}) =$ &c.

5695. (By Professor EVANS, M.A.)—Find the locus of a point the sum of the squares of whose distances from the vertices of a given triangle is constant.

Solution by H. MURPHY; R. KNOWLES, L.C.P.; J. O'REGAN; *and others.*

Let D be a point in the locus: bisect AB in F, join FD and FC. Because $\qquad AD^2+DB^2 = 2AF^2+2FD^2,$



we have

$$AD^2+DB^2+DC^2 = 2AF^2+2FD^2+DC^2,$$

which is given; but $2AF^2$ is given, therefore $2FD^2+DC^2$ is given, and base FC of triangle FDC is given; hence locus of D is a circle; which can easily be constructed.

This is a particular case of the more general question, "Given base, and multiple squares of lines forming a point to the vertices of a triangle," which may be solved as follows:—Divide base such that $mAF = nBF$, then $\qquad mAD^2+nDB^2 = mAF^2+nFB^2+(m+n)FD^2,$

$\therefore \quad mAD^2+nDB^2+pDC^2 = mAF^2+nFB^2+pDC^2+(m+n)FD^2$, is given; but mAF^2+nFB^2 is given, therefore $(m+n)FD^2+pDC^2$ is given, and the base FC of $\triangle$FDC is given; hence locus of D is a circle, which can be constructed by dividing FB such that p times one segment $= (m+n)$ the other segment, then DE becomes known, &c.

5565. (By Professor TOWNSEND, F.R.S.)—In axial refraction through a system of any number of ordinary media bounded by spherical surfaces of any radii having their centres ranged at any distances along a common axis, if P and Q be the positions of the two principal foci, and X and Y those of any pair of conjugate foci, on the axis of the system; show that the rectangle PX . QY is constant, in magnitude and sign, for all positions of X and Y.

Solution by the PROPOSER.

Since, in refraction, at every sphere of separation of two contiguous media, the two foci before and after refraction cut in a constant anharmonic ratio, equal to that of the refractive indices of the media, and there-

fore divide homographically the radius of the sphere, the first and last foci of the entire refraction, consequently, divide homographically the axis of the system; hence, if X and Y, X′ and Y′ be any two pairs of conjugate foci, and P and Q as above the two principal foci, of the entire refraction,

since $\{XX'P\infty\} = \{YY'\infty Q\} = \{Y'YQ\infty\}$,

therefore $PX : PX' = QY' : QY$, or $PX . QY = PX' : QY'$;

and therefore, &c.

5701. (By R. F. Scott, M.A.).—Prove that

$$I \equiv \int_0^\infty \frac{\sin x}{1-\sin^2\alpha \sin^2 x} \cdot \frac{dx}{x} = \tfrac{1}{2}\pi \sec\alpha.$$

Solution by R. Knowles, L.C.P.; H. Stabenow; *and others.*

$$\frac{\sin x}{(1-\sin^2\alpha . \sin^2 x)\, x} = \frac{\sin x}{x} + \frac{\sin^2\alpha . \sin^3 x}{x} + \frac{\sin^4\alpha . \sin^5 x}{x} + \dots\dots$$

Now, $\displaystyle\int_0^\infty \frac{\sin rx\, dx}{x} = \tfrac{1}{2}\pi$ (Todhunter's *Integral Calculus*, Art. 290);

$$\int_0^\infty \frac{\sin^2\alpha . \sin^3 x}{x} = \sin^2\alpha \left(\int_0^\infty \frac{3}{4} \cdot \frac{\sin x}{x} - \int_0^\infty \frac{\sin 3x}{4x} \right) = \tfrac{1}{2} . \sin^2\alpha . \tfrac{1}{2}\pi,$$

$$\int_0^\infty \frac{\sin^4\alpha . \sin^5 x}{x} = \tfrac{3}{8} \cdot \sin^4\alpha . \tfrac{1}{2}\pi, \quad \text{and so on;}$$

$$\therefore \int_0^\infty \frac{\sin x}{1-\sin^2\alpha . \sin^2 x} \cdot \frac{dx}{x} = \tfrac{1}{2}\pi \left(1 + \tfrac{1}{2}\sin^2\alpha + \tfrac{3}{8}\sin^4\alpha + \dots\right) = \tfrac{1}{2}\pi u \text{ suppose.}$$

Put $y = \sin\alpha$, then, using Boole's symbolical method, we have

$$Du - (D-1)\epsilon^{2\theta} u = 0 \quad \text{or} \quad (1-y^2)\frac{du}{dy} = yu,$$

$$\therefore \int \frac{du}{u} = \int \frac{y\,dy}{1-y^2} \log u = \log \frac{1}{(1-y^2)^{\frac{1}{2}}}; \;\therefore u = \frac{1}{(1-y^2)^{\frac{1}{2}}} = \sec\alpha;$$

therefore we obtain, finally, $I = \tfrac{1}{2}\pi \sec\alpha$.

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141. Notes on the Metrical System. *By the* Rev. F. D. Thomson, M.A.

I wish to call attention to some discrepancies I have noticed in certain standard books with respect to the equivalents of the measures of capacity in the English and French systems.

It may be as well to premise that the *data* are—that the English yard is the length of a certain bar of metal at 62° F., that the metre is the length of a certain bar of platinum at 32° F., and that a gallon is ten times the volume of a mass of pure water which balances in air of stated density, at a stated temperature, a particular mass of metal called the standard pound. As the result of observation it is found that one metre $=39\cdot37079$ inches, and that one gallon equals, according to different observers, from 274·271 to 274·276 cubic inches. Everything else, so far as regards measures of capacity, is a matter of *calculation*. It follows at once that 1 litre, *i.e.*, a cubic decimetre $= \dfrac{(\text{number of inches in a decimetre})^3 \times 8}{\text{number of cubic inches in a gallon}}$ pints.

Now in MILLER's *Elements of Chemistry* the gallon is taken as 277·276 cubic inches, while it is stated that 1 litre = 61·024 cubic inches or 1·765 pints (in the text), or 1·76377 pints (in the tables at the end). The value 61·024 has been obtained by taking 3·937 inches for the decimetre: if the other decimal figures are used, we get 61·027 cubic inches as the true value of the litre. Again, with the given values, 1 litre should be $\dfrac{61\cdot024 \times 8}{277\cdot276}$ pints or 1·76067 pints.

The value assigned in the Act of Parliament of 1864 is 1·76077 pints, from which it follows, taking the litre as 61·027 cubic inches, that the gallon was taken as 277·274 cubic inches. Similar discrepancies are to be found in the logarithmic formulæ for conversion given by MILLER.

In CHISHOLM's *Weighing and Measuring*, the gallon is taken at 277·274 cubic inches, while the litre is said to be ·88072 quarts or 1·76144 pints, and its equivalent, 1000 cubic centimetres, is reckoned at 61·08 cubic inches. These are easily seen to be inconsistent. The 61·08 has been obtained, I believe, by cubing 3·938, which is stated to be the value in *inches* of the decimetre *when the bar upon which it is marked is raised in temperature from* 32° *F. to* 62° *F.* This is, however, misleading, as the measure of capacity should depend solely upon the standard of length at its own standard temperature. How the ·88072 quarts is obtained I do not know.

I notice also, in MERRIFIELD's *Technical Arithmetic and Mensuration*, that the gallon is taken as 277·274 cubic inches and the litre as 61·024 cubic inches, while the gallon is said to be 4·54102 litres, and the litre to be ·220215 gallons or 1·76172 pints. These numbers also are inconsistent. Taking the correct value of the litre and 277·274 cubic inches for the gallon, the true equivalents are 4·54346 litres and ·220096 gallon.

Of course for practical work the differences I have noticed are of no importance, but it seems absurd to give results to five or six decimal places, if the accuracy of the *third* place cannot be depended on.

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142. NOTE ON THE SOLUTION OF A CONGRUENCE OF THE FIRST DEGREE WHEN THE MODULUS IS A COMPOSITE NUMBER. *By* CHRISTINE LADD.

According to the solution given by SERRET (*Cours d'Algèbre Supérieure, quatrième édition*, Art. 289), the root of the congruence

$$ax + b \equiv 0 \pmod{M},$$

where $M = M_1M_2M_3 \dots M_k$, is obtained by determining α, α_1, α_2 from the congruences $a\alpha + b \equiv 0 \pmod{M_1}$, $a\alpha_1 + b_1 \equiv 0 \pmod{M_2}$,

$$a\alpha_2 + b_2 \equiv 0 \pmod{M_3}, \quad \dots\dots,$$

in which $$b_1 = \frac{a\alpha + b}{M_1}, \quad b_2 = \frac{a\alpha_1 + b_1}{M_2}, \quad \dots\dots,$$

and writing $$x = \alpha + M_1\alpha_1 + M_1M_2\alpha_2 + M_1M_2M_3\alpha_3 + \dots \quad \dots\dots\dots\dots (m).$$

But if we determine z, z_1, z_2, ... from the congruences

$$az + 1 \equiv 0 \pmod{M_1}, \quad az_1 + 1 \equiv 0 \pmod{M_2},$$
$$az_2 + 1 \equiv 0 \pmod{M_3}, \quad \dots\dots,$$

we have $$bz = \alpha, \quad b_1z_1 = \alpha_1, \quad b_2z_2 = \alpha_2, \quad \dots\dots;$$

or, substituting for b_1, b_2, ... their values as given above,

$$\alpha_1 = \frac{a\alpha + b}{M_1} z_1 = \frac{b(az+1)}{M_1} z_1, \quad \alpha_2 = \frac{a\alpha_1 + b_1}{M_2} z_2 = \frac{b(az+1)(az_1+1)}{M_1M_2} z_2,$$
$$\alpha_3 = \frac{b(az+1)(az_1+1)(az_2+1)}{M_1M_2M_3} z_3, \quad \dots\dots.$$

Substituting these values of α, α_1, α_2 ... in the equation (m), we have

$$\chi = b[z + (az+1)z_1 + (az+1)(az_1+1)z_2 + (az+1)(az_1+1)(az_2+1)z_3 + \dots]$$
$$= b[\Sigma(z) + a\Sigma(zz_1) + a^2\Sigma(zz_1z_2) + \dots],$$

where $\Sigma(zz_1z_2)$ equals the sum of the products of the z's taken three by three, &c.

This solution differs from that of SERRET in the fact that the congruences to be solved are of a more simple form, and that the calculation of b_1, b_2, ... is rendered unnecessary. It is not, however, easier of application when the numbers are large.

It is not always desirable to factor a composite modulus. Is there any way of determining when it is desirable and when not?

5700. (By Colonel CLARKE, C.B., F.R.S.)—Required an investigation of the following problem:—You can ascertain whether a boiled egg is hard-boiled or soft by this process,—Lay the egg on a smooth horizontal surface; spin it (initially round a vertical axis perpendicular to its axis of figure); then, if soft, it continues to rotate in that way (Fig. 1); but if hard, it soon gets up on end thus (as shown in Fig. 2).

FIG. 1.

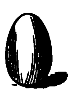


FIG. 2.

[Col. CLARKE remarks that any mathematician can try the experiment at breakfast, adding the necessary proviso that it is well not to break the egg.]

Solution by SEPTIMUS TEBAY, B.A.

Mr. ORCHARD's solution of this problem* seems to be somewhat defective. He says, "If the contents are liquid there is no vertical force produced upon the centre of gravity." Now, whether the egg be at rest or in

* See page 30 of this volume of the *Reprint*.

motion, no force acting on the centre of gravity can turn the egg about this or any other point. Suppose the egg to be filled with incompressible fluid, and free from friction. When twirled about a vertical axis through the centre of gravity, no rotatory motion will be communicated to the fluid; and, therefore, there will be no alteration of position, except what is due to the mass of the shell. This condition of itself would form an interesting subject of investigation.

Again, Mr. ORCHARD says, "But if the contents are *solid*, we shall have to consider a resultant force made up of the (horizontal) spinning force, and of the moment of inertia due to rotation about the centre of gravity." What is "horizontal spinning force"? Is not the moment of inertia also horizontal?

This problem, when the egg is any solid of revolution, has been completely solved by Mr. WOOLHOUSE in the *Gentleman's Diary* for 1838. The fact, as stated, may be inferred from general reasoning; for it is a principle of the free motion of a body about an axis through the centre of gravity that there is a tendency to rotate about a principal axis, and nature prefers that about which the moment of inertia is a maximum. But the axis of the egg not being quite horizontal when twirled about a vertical axis through the centre of gravity, this line will seek a different position, and the centre remaining in the initial vertical line, the perpendicular from this point on the plane will no longer pass through the point of contact; consequently there will be motion about this point, and, as the centre of gravity seeks the lowest position, the smaller end of the egg will be elevated.

5598. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_0^a \frac{\sin^{n-2}\theta\,(n-1-n\sin^2\theta-x\sin\theta)\,d\theta}{(1+x\sin\theta)^{n+1}} = \frac{\cos\alpha\sin^{n-1}\alpha}{(1+x\sin\alpha)^n},$$

if $1+x\sin\alpha$ be positive, and n be any whole number. [Professor WOLSTENHOLME believes that, when $1+x\sin\alpha$ is *negative*, the equation is still true if the principal value of the integral be taken, though he states that he has not investigated this point closely.]

Solution by J. HAMMOND, M.A.; C. B. S. CAVALLIN; *and others.*

Calling $\dfrac{\cos\theta\sin^{n-1}\theta}{(1+x\sin\theta)^n} = u$, we have

$$\frac{du}{d\theta} = -\frac{\sin^n\theta}{(1+x\sin\theta)^n} + \frac{(n-1)\sin^{n-2}\theta\cos^2\theta}{(1+x\sin\theta)^n} - \frac{nx\cos^2\theta\sin^{n-1}\theta}{(1+x\sin\theta)^{n+1}}$$

$$= \frac{\sin^{n-2}\theta}{(1+x\sin\theta)^{n+1}}\left\{(n\cos^2\theta-1)(1+x\sin\theta)-nx\cos^2\theta\sin\theta\right\}$$

$$= \frac{\sin^{n-2}\theta\,(n-1-n\sin^2\theta-x\sin\theta)}{(1+x\sin\theta)^{n+1}}.$$

Thus $$\int_0^{\alpha} \frac{\sin^{n-2}\theta\,(n-1-n\sin^2\theta - x\sin\theta)\,d\theta}{(1+x\sin\theta)^{n+1}} = \frac{\cos\alpha\,\sin^{n-1}\alpha}{(1+x\sin\alpha)^n}.$$

This result holds when the expression under the integral does not become infinite for values of θ from $\theta = 0$ to $\theta = \alpha$ inclusive.

That is, n may be any positive quantity not < 1, and $1 + x\sin\theta$ is of invariable sign, or α being supposed $< \frac{1}{2}\pi$, $1 + x\sin\alpha$ is positive.

5638. (By J. J. WALKER, M.A.)—Prove that the curve

$$2(n-1)\frac{du}{dx}\frac{du}{dy}\frac{du}{dz} - x\frac{du}{dy}\frac{du}{dz}\frac{d^2u}{dx^2} - y\frac{du}{dz}\frac{du}{dx}\frac{d^2u}{dy^2} - z\frac{du}{dx}\frac{du}{dy}\frac{d^2u}{dz^2} = 0$$

passes through the points of inflexion on $u = 0$, n being the order of u.

Solution by the PROPOSER; Prof. EVANS, M.A.; *and others.*

Multiplying by $(n-1)^2$, and substituting for $(n-1)\frac{du}{dx}$, $x\frac{d^2u}{dx^2} + y\frac{d^2u}{dx\,dy} + z\frac{d^2u}{dz\,dx}$, and similarly for $(n-1)\frac{du}{dy}$, $(n-1)\frac{du}{dz}$, the result reduces identically to

$$xyz(-abc - 2fgh + af^2 + bg^2 + ch^2)$$
$$-(ghx + hfy + fgz)(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy),$$

where $$a = \frac{d^2u}{dx^2}, \;\ldots\ldots\; h = \frac{d^2u}{dx\,dy},$$

that is, to $$xyz\mathrm{H}u + (ghx + hfy + fgz)\,u = 0.$$

5614. (By W. H. H. HUDSON, M.A.)—If r be the radius vector of a point P on a curve corresponding to a point of inflexion on the inverse with respect to the origin, prove that the tangent at P is inclined to the radius vector at an angle $\sin^{-1}\frac{r}{2\rho}$, where ρ is the radius of curvature at P.

Solution by W. J. C. SHARP, B.A.; R. F. DAVIS, B.A.; *and others.*

Let Q be the point of inflexion on the inverse, O the centre of inversion, and C the centre of curvature at P; then, since the circles of curvature at P and Q are (WILLIAMSON'S *Diff. Calc.*, Art. 227) inverses with respect to O; and as Q is a point of inflexion the circle of curvature has an infinite

radius, and is in fact the inflexional tangent, therefore O must be a point on the circle of curvature at P. If PT be the tangent at P,

$$\sin \mathrm{OPT} = \cos \mathrm{CPO} = \cos \mathrm{COP} = \frac{r}{2\rho}, \text{ therefore } \mathrm{OPT} = \sin^{-1}\frac{r}{2\rho}.$$

[If (r, θ) be the polar coordinates of P, and (ω, ω') the inclinations of the tangents at P and the corresponding point on the inverse to the initial line; then, since these tangents are equally inclined to the common radius vector, we have $\omega' = \omega - 2\theta$; hence, when

$$d\omega' = 0, \ \frac{d\omega}{d\theta} = 2; \ \text{or} \ \frac{rd\theta}{ds} = \frac{r}{2} \cdot \frac{d\omega}{ds} = \frac{r}{2\rho}.]$$

5407. (By S. Roberts, M.A.)—A system of conics has a common focus and directrix, another system also has a common focus and directrix; required the locus of the intersection of corresponding conics having equal eccentricities.

I. *Solution by* H. T. Gerrans, B.A.; Prof. Evans, M.A.; *and others.*

Let $(x_1y_1) \equiv x\cos\alpha + y\sin\alpha - p_1 = 0$, $(x_2y_2) \equiv x\cos\beta + y\sin\beta - p_2 = 0$, be the foci and directrices of the two systems. Then, if e be the common eccentricity of a pair of corresponding conics, their equations are

$$(x-x_1)^2 + (y-y_1)^2 = e^2(x\cos\alpha + y\sin\alpha - p_1)^2,$$
$$(x-x_2)^2 + (y-y_2)^2 = e^2(x\cos\beta + y\sin\beta - p_2)^2.$$

Eliminating e, we get as the equation of the locus of their intersections, which passes through seven fixed points, (1) the point of intersection of the directrices; (2), (3), (4), (5), the points (imaginary) where each directrix cuts the point circle given by the other focus; (6), (7), the points of intersection of these point circles.

II. *Solution by the* Proposer; E. Rutter; *and others.*

Referring the conics to two sets of rectangular axes at the foci, the equations of corresponding conics are

$$x^2 + y^2 = e^2\gamma^2, \ x'^2 + y'^2 = e^2\gamma'^2; \ \text{hence} \ (x^2+y^2)\gamma'^2 = (x'^2+y'^2)\gamma^2,$$

a curve of the 4th degree.

A system of conics has a common centre, and their axes have the same directions. There are also two similar systems about two other points as centres; required the locus of the intersections of corresponding conics of the three systems having equal axes.

Referring the conics to three sets of Cartesian coordinates (rectangular), with the respective centres as origin, we have

$$\left.\begin{aligned} Ax^2 + By^2 + C &= 0 \\ Ax'^2 + By'^2 + C &= 0 \\ Ax''^2 + By''^2 + C &= 0 \end{aligned}\right\} \ \text{or} \ \begin{vmatrix} x^2, & y^2, & 1 \\ x'^2, & y'^2, & 1 \\ x''^2, & y''^2, & 1 \end{vmatrix} = 0$$

is the locus required of the 4th degree.

5310. (By Professor LLOYD TANNER, M.A.)—If $n(n-1) = kr$, $n < r$; prove that $2n-1$ is prime to r, all the letters representing positive integers.

Solution by the PROPOSER.

This is a particular case of the obvious theorem that, if a, b be relatively prime, then $a+b$ is prime to ab.

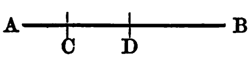
143. ON MR. WOOLHOUSE'S THEORY OF PROBABILITY.
By ARTEMAS MARTIN, M.A.

My object in preparing this paper is not to advocate or defend any pet theory of my own, but to bring together a few solutions bearing upon the disputed point, and leave the decision to wiser heads.

1. The following problem (*Diary*, Quest. 1912) was proposed in the *Lady's and Gentleman's Diary* for 1856, and solved in the succeeding number for 1857 by Mr. SEPTIMUS TEBAY :—

"A plank is cut at random into three lengths or rectangular pieces. If they be simultaneously placed upon one another at random (like three bricks in a wall), the chance that they will not fall down is $\frac{7}{2} - \frac{1}{3}\pi^2$."

The substance of Mr. TEBAY's solution is as follows :—



Let AB represent the plank; put $AB = a$, $AC = x$, $AD = y$; then the pieces are x, $y-x$, and $a-y$. The chance that the centre of gravity of the piece DB falls upon CD is $\dfrac{y-x}{(y-x)+(a-y)} = \dfrac{y-x}{a-x}$; the chance that the centre of gravity of the pieces DB and CD falls on AC is $\dfrac{x}{x+(y-x)} = \dfrac{x}{y}$. Hence the probability that the three pieces are in equilibrium is

$$\frac{y-x}{a-x} \times \frac{x}{y} = \frac{x(y-x)}{y(a-x)}.$$

The limits of y are x and a; of x, 0 and a; and the chance required is

$$p = \int_0^a \int_x^a \frac{x(y-x)\,dy\,dx}{y(a-x)} \div \int_0^a \int_x^a dy\,dx$$

$$= \frac{2}{a^2} \int_0^a \int_x^a \frac{x(y-x)\,dy\,dx}{y(a-x)} = \frac{2}{a^2} \int_0^a \frac{x}{a-x} \left(a - x + x \log \frac{x}{a}\right) dx$$

$$= 1 + \frac{2}{a^2} \int_0^a \frac{x^2}{a-x} \log\left(\frac{x}{a}\right) dx = \frac{7}{2} - \frac{\pi^2}{3}.$$

This solution does not appear to be in accordance with Mr. WOOLHOUSE'S

formula, "$p = \dfrac{\text{total favourable cases}}{\text{total all cases}}$," [see *Reprint*, Vol. XXVII., p. 79,] according to which the chance should, it seems to me, be

$$p = \int_0^a \int_x^a x\,(y-x)\,dy\,dx \div \int_0^a \int_x^a y\,(a-x)\,dy\,dx\,; \quad \text{or if not, why not?}$$

2. In the *Diary* for 1861, p. 62, Mr. Woolhouse gives a solution of the following problem :—

"An unknown cone being cut by an unknown plane, determine the probability of the section being an ellipse." [*Diary*, Quest. 1970.]

Putting d for the semi-angle of the cone, Mr. Woolhouse finds the chance for a cone of definite extent to be

$$p = \frac{(\frac{1}{2}\pi - a)\sin a + \cos a \cos 2a}{(\frac{1}{2}\pi - a)\sin a + \cos a}.$$

To get the chance when the cone is unknown, he multiplies numerator and denominator by da, then integrates from 0 to $\frac{1}{2}\pi$, and finds $p = \dfrac{\pi - \frac{4}{3}}{\pi + 4}$. This solution conforms to the method he approves in the *Reprint*, referred to above. Are these solutions correct?

3. In the first number of the *Mathematical Visitor*, I proposed the following problem :—

"The first of two casks contains a gallons of wine, and the second b gallons of water; part of the water is poured into the first cask, and then part of the mixture is poured back into the second; find the probability that not more than one-n^{th} of the contents of the second cask is wine."

Mr. Henry Heaton solved this problem in substance as follows :—

Let x = number of gallons of water poured into the first cask, and y = number of gallons of the mixture poured back into the second. Then y must not exceed $\dfrac{(a+x)(b-x)}{(n-1)a-x}$, the probability of which is $\dfrac{y}{a+x}$ or $\dfrac{b-x}{(n-1)a-x}$, and the probability required is

$$p = \int_0^b \left(\frac{y}{a+x}\right) dx \div \int_0^b dx = \frac{1}{b}\int_0^b \frac{(b-x)\,dx}{(n-1)a-x}$$
$$= 1 - \frac{(n-1)a-b}{b}\log\left(\frac{(n-1)a}{(n-1)a-b}\right).$$

According to Mr. Woolhouse's first formula in the *Reprint*, which, he says, is universally applicable, the solution would be

$$p = \int_0^b y\,dx \div \int_0^b (a+x)\,dx = \int_0^b \frac{(a+x)(b-x)\,dx}{(n-1)a-x} \div \int_0^b (a+x)\,dx$$
$$= \frac{2na-b}{2a+b} - \frac{2na\,[(n-1)a-b]}{2ab+b^2}\log\left(\frac{(n-1)a}{(n-1)a-b}\right). \quad \text{Which is correct?}$$

[Mr. Martin having requested the Editor to obtain for him the opinions, at length, of Mr. Woolhouse, Mr. Stephen Watson, Mr. Tebay, Professor Crofton, and others,—" wiser heads," as he calls them,—upon this "important and disputed point in the Theory of Probability," we

forwarded a proof of the foregoing article, with the author's request, to each of the mathematicians named.

In answer to the request, Mr. TEBAY remarked that he has still some recollection of the principle which guided him in the solution of his "plank" problem more than twenty years ago. After obtaining the chances quoted by Mr. MARTIN, he observed that the probabilities that the points of division fell where they were assumed to fall, were $\frac{dx}{a}$ and $\frac{dy}{a}$; and these four being independent, the probability of their concurrence was of course

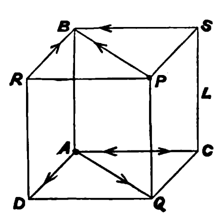
$$\frac{y-x}{a-x} \cdot \frac{x}{y} \cdot \frac{dx}{a} \cdot \frac{dy}{a}.$$

This he took to be the differential of the varying probability required; but on examination it does not appear to be satisfactory, since the total favourable and unfavourable cases can be separately obtained. These are $\frac{1}{24}$ and $\frac{1}{8}$, and the chance simply $\frac{1}{3}$. For the same reasons, the second solution to Mr. MARTIN's third problem is decidedly the most satisfactory. With regard to the second problem no objection can be raised, as Mr. WOOLHOUSE has followed the legitimate principle *ab initio*.]

5524. (By Professor TOWNSEND, F.R.S.)—A rigid body being supposed to revolve round a fixed axis; prove the following construction for the position of the axis of the wrench, to which the centrifugal forces of its several elements are reducible in their canonical form:—

On the plane connecting the axis of rotation with the centre of inertia of the body, project orthogonally the rectangular hyperbola locus of the principal points of all parallels to the axis of rotation which are principal axes of the body. The perpendicular to the axis, in the plane of connexion, through its near point of intersection with the projected hyperbola, will be the required position of the axis in question.

Solution by the PROPOSER.



Let L (see Fig.) be the axis of rotation, A the centre of inertia of the body, AB and AC the parallel and perpendicular through A to L in the plane AL, AD the perpendicular through A to the plane AL, ABPQ the plane through AB locus of all the principal axes of the body which are parallel to L, PQ the intersection of that plane with the plane through L perpendicular to AL, P the principal point of the principal axis PQ, and PQ, PR, PS the three edges opposite to AB, AC, AD of the rectangular parallelepiped of which A and P are a pair of opposite vertices and AP the connecting diagonal. Then, the locus of P being, as is well known, a rectangular hyperbola in the plane ABPQ, of which AB and AQ are the asymptotes, and the point S on the axis of rotation L being the orthogonal projection of P on the plane AL, it is to be shewn that the right

line SB, perpendicular to L in the plane AL, is the axis of the wrench to which the centrifugal forces of the several elements of the rotating body are reducible in their canonical form.

Denoting by m the mass of the body, and by ω the velocity of its rotation round L; since, on well-known dynamical principles, the centrifugal effect of the rotation round CS, with velocity ω, is equivalent to that of a rotation round AB with equal velocity, combined with a single force $\omega^2 . m . \mathrm{CA}$ passing through A and acting from C towards A, while that of the rotation round AB is equivalent to the pair of equal and opposite forces $\omega^2 . m . \mathrm{AQ}$ and $\omega^2 . m . \mathrm{PB}$ acting from A to Q and from P to B respectively, or, by resolution, to the two pairs of equal and opposite forces $\omega^2 . m . \mathrm{AC}$ and $\omega^2 . m . \mathrm{SB}$ acting from A to C and from S to B respectively, and $\omega^2 . m . \mathrm{AD}$ and $\omega^2 . m . \mathrm{RB}$ acting from A to D and from R to B respectively; and since, of the aforesaid five forces, two of equal magnitude, $\omega^2 . m . \mathrm{CA}$ and $\omega^2 . m . \mathrm{AC}$, destroy each other by direct opposition, the centrifugal effect of the rotation round CS is consequently equivalent to the simple force $\omega^2 . m . \mathrm{SB}$ acting from S to B along the line SB, combined with the pair of equal and opposite forces $\omega^2 . m . \mathrm{AD}$ and $\omega^2 . m . \mathrm{RB}$ acting from A to D and from R to B in the plane ABRD perpendicular to the line SB, that is, to a wrench of which SB is the axis, and therefore &c.

5392. (By Professor EVANS, M.A.)—If $\frac{p_n}{q_n}$ be the last convergent in the first period of $\mathrm{A}^{\frac{1}{2}}$ expanded as a continued fraction, and r the greatest integer contained in $\mathrm{A}^{\frac{1}{2}}$, show that $p_n = rq_n + q_{n-1}$.

Solution by J. W. SHARPE, M.A.; Prof. NASH, M.A.; *and others.*

If we expand $\mathrm{A}^{\frac{1}{2}}$ in a continued fraction we shall get results of the form

$$x = \mathrm{A}^{\frac{1}{2}} = a + \frac{1}{x_1}; \quad x_1 = \frac{\mathrm{A}^{\frac{1}{2}} + \mathrm{E}_1}{\mathrm{D}_1} = a_1 + \frac{1}{x_2}; \ \ldots\ldots$$

$$x_{n-1} = \frac{\mathrm{A}^{\frac{1}{2}} + \mathrm{E}_{n-1}}{\mathrm{D}_{n-1}} = a_{n-1} + \frac{1}{x_n}; \quad x_n = \frac{\mathrm{A}^{\frac{1}{2}} + \mathrm{E}_n}{\mathrm{D}_n} = a_n + \frac{1}{x_{n+1}};$$

where $a, a_1, \ldots$ are the successive quotients, and $x, x_1, \ldots$ the successive complete quotients. Now we have

$$x = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}}, \quad \text{or} \quad \mathrm{A}^{\frac{1}{2}} = \frac{p_n (\mathrm{E}_{n+1} + \mathrm{A}^{\frac{1}{2}}) + \mathrm{D}_{n+1} p_{n-1}}{q_n (\mathrm{E}_{n+1} + \mathrm{A}^{\frac{1}{2}}) + \mathrm{D}_{n+1} q_{n-1}};$$

therefore $(q_n \mathrm{E}_{n+1} + \mathrm{D}_{n+1} q_{n-1} - p_n) \mathrm{A}^{\frac{1}{2}} + q_n \mathrm{A} - p_n \mathrm{E}_{n+1} - \mathrm{D}_{n+1} p_{n-1} = 0.$

But $\mathrm{A}^{\frac{1}{2}}$ is an irrational quantity; hence we have

$$p_n = q_n \mathrm{E}_{n+1} + \mathrm{D}_{n+1} q_{n-1}, \quad \text{and} \quad q_n = \frac{p_n \mathrm{E}_{n+1} + \mathrm{D}_{n+1} p_{n-1}}{\mathrm{A}};$$

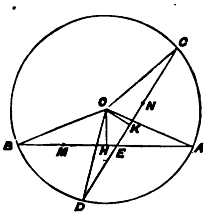
but $\mathrm{E}_{n+1} = a = r, \quad \mathrm{D}_{n+1} = 1; \quad$ therefore $p_n = rq_n + q_{n-1};$

and also $$q_n = \frac{rp_n + p_{n-1}}{\mathrm{A}}.$$

5450. (By Hugh McColl, B.A.)—If two points are taken at random inside a circle, and a random line be drawn through each, find the chance that the lines will intersect within the circle.

Solution by E. B. Seitz.

Let M and N be the random points, AB and CD the chords formed by the random lines through M and N, E their intersection, O the centre of the circle, OH and OK the perpendiculars on AB and CD. Now, while M is fixed in position, and AB in direction, N may range over the circle, the direction of CD remaining the same; then CD may take all possible directions; next M may range over the circle, the direction of AB remaining unchanged; and lastly, AB may take all possible directions.



Let $OA = r$, $\angle AOH = \theta$, $\angle AEC = \phi$, $\angle COK = \psi$, and $\omega =$ the angle AB makes with some fixed line.

Then that the chords AB and CD will intersect, the limits of ψ must be $\theta-\phi$ and $\theta+\phi$ when $\phi<\theta$, and $\phi-\theta$ and $\phi+\theta$ when $\phi>\theta$. The limits of ϕ are 0 and $\frac{1}{2}\pi$, and doubled; of θ, 0 and $\frac{1}{2}\pi$, and doubled; and of ω, 0 and π. Hence, since the whole number of ways the two lines can be drawn through the points is $\pi^4 r^4$, the required chance is

$$\frac{4}{\pi^4 r^4}\int_0^{\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi} 2r^2\sin^2\psi\, d\phi\, d\psi + \int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta} 2r^2\sin^2\psi\, d\phi\, d\psi\right\} d\omega \,.\, 2r^2\sin^2\theta\, d\theta$$

$$=\frac{8}{\pi^4}\int_0^{\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}(2\phi-\cos 2\theta\sin 2\phi)\,d\phi+\int_{\theta}^{\frac{1}{2}\pi}(2\theta-\sin 2\theta\cos 2\phi)\,d\phi\right\}\sin^2\theta\, d\omega\, d\theta$$

$$=\frac{4}{\pi^4}\int_0^{\pi}\int_0^{\frac{1}{2}\pi}(2\pi\theta-2\theta^2+1-\cos 2\theta)\sin^2\theta\, d\omega\, d\theta=\frac{1}{3}+\frac{5}{2\pi^2}.$$

5582. (By E. B. Seitz.)—A line is drawn at random across a convex polygon; show that its average length is $\pi s^{-1}\Delta$, where s is the perimeter of the polygon, and Δ its area.

Solution by the Proposer.

Let ABC ... be the polygon, AM a fixed line cutting the polygon, $x =$ the length of the random line, $\theta =$ the angle it makes with AM, $y =$ its distance from the most remote vertex of the polygon on the side from which θ is measured, $z =$ the normal range of the line, $a, b, c \dots =$ the sides of the polygon, $\alpha, \beta, \gamma, \dots =$ the angles which AM makes with the sides, $\phi_1, \phi_2, \phi_3, \dots =$ the angles which the random line makes with the sides, these angles all increasing with θ. Then we have

$$z = \tfrac{1}{2}(a\sin\phi_1 + b\sin\phi_2 + c\sin\phi_3 + \dots)$$

the sum of all the lines that can be drawn across the polygon is

$$\int_0^\pi \int_0^z x \, d\theta \, dy = \Delta \int_0^\pi d\theta = \pi\Delta,$$

and the whole number of lines is

$$\int_0^\pi z\,d\theta = \tfrac{1}{2}a\int_a^\pi \sin\phi_1 d\phi_1 + \tfrac{1}{2}a\int_0^a \sin\phi_1 d\phi_1 + \tfrac{1}{2}b\int_\beta^\pi \sin\phi_2\,d\phi_2 + \tfrac{1}{2}b\int_0^\beta \sin\phi_2\,d\phi_2$$

$$+ \tfrac{1}{2}c\int_\gamma^\pi \sin\phi_3\,d\phi_3 + \tfrac{1}{2}c\int_0^\gamma \sin\phi_3\,d\phi_3 + \ldots = a+b+c+\ldots = s.$$

Hence the average length of the line is $\pi s^{-1}\Delta$.

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144. PROOF OF THE THEOREM THAT THE SURFACE-INTEGRAL OF A FLUX IS EQUAL TO THE LINE-INTEGRAL OF A FLOW, THE LINE-INTEGRAL BEING TAKEN ROUND THE CLOSED LINE WHICH BOUNDS THE SURFACE.

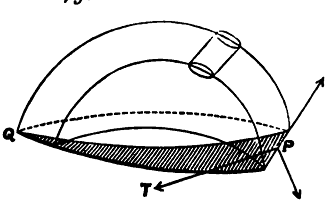
If v and w are continuous functions through a closed space, then

$$\iiint \left(\frac{dv}{dy}\frac{dw}{dx} - \frac{dv}{dx}\frac{dw}{dy}\right) dx\,dy\,dz$$

$$\equiv \iint w\left(l\frac{dv}{dy} - m\frac{dv}{dx}\right) dS - \iiint w\left(\frac{d^2v}{dx\,dy} - \frac{d^2v}{dy\,dx}\right)$$

$$\equiv \iint w\left(l\frac{dv}{dy} - m\frac{dv}{dx}\right) dS,$$

where l, m are the cosines of the angles which the normal to the surface bounding the space makes with the axes of x, y.



Let v be the function which expresses the length of the normal drawn from (x, y, z) to the surface, then $\frac{dv}{dx}$ is the x direction-cosine of the outward normal; and take as the closed space through which the integration is made the space bounded by the given surface, a parallel surface at distance $\delta\tau$, and the band PQ. Then

$$\iiint \left(m\frac{dw}{dx} - l\frac{dw}{dy}\right) d\sigma\,.\,d\tau = -\iint w\,.\,\frac{dz}{ds}\,.\,ds\,.\,d\tau.$$

The part of the double integral vanishing everywhere except along the band. And, $\frac{dz}{ds}$ being $= m\frac{dv}{dx} - l\frac{dv}{dy}$, the z direction-cosine of the line per-

pendicular to the normal to the surface and the normal to the band, *i.e.*, the z direction-cosine of the tangent line to the curve PQ..

Hence $$\iint\left(m\frac{dw}{dx}-l\frac{dw}{dy}\right)d\sigma=-\int w\frac{dz}{ds}ds,$$

or $$\iint\left[l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right]d\sigma$$
$$=\int(u\,dx+v\,dy+w\,dz).$$

[The author of the foregoing proof (who has omitted to send his name and address) states that he understands there is a new proof of the proposition, different from his own, coming out in the forthcoming edition of THOMSON and TAIT'S *Natural Philosophy*. He adds furthermore, that by taking $w=r^{-1}$, we get the usual form for the vector potential of a magnetic shell at a point P, and remarks that the proof in Art. 24 of MAXWELL'S *Electricity* is rather difficult.]

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145. ON THE MATHEMATICAL QUESTION, WHAT IS A TREE?

By Professor SYLVESTER, F.R.S.

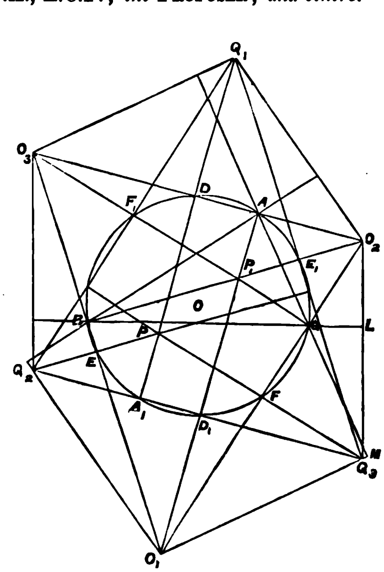
A tree is a system of points and lines in which every line is limited by two of the points, and every point is connected with every other by a single line or single series of lines; or, more generally, a tree is a system of ideas in which every idea is related directly or mediately *in only one way* to every other, two ideas being said to be mediately related when they may be regarded as the extremes of a chain of ideas, capable of being so taken in succession as that each non-extreme shall stand in direct relation to its immediate antecedent and consequent.

5617. (By S. CONSTABLE.)—If O, O_1, O_2, O_3 are the circumscribed and escribed centres of a triangle, and perpendiculars are drawn from O_1, O_2, O_3 on the sides, meeting two and two and forming another triangle $Q_1Q_2Q_3$; then, considering the two triangles $O_1O_2O_3$ and $Q_1Q_2Q_3$, prove that—(1) they are in perspective, the centre of perspective being the point O; (2) the perpendiculars from the vertices of either on the sides of the other meet in a point, the two points being the centres of the circumscribing circles of the two triangles; (3) they have the same nine-point circle; (4) they are so placed that any point connected with the one, and

the corresponding point of the other, are equally equidistant from O and in the same straight line with O; (5) the area of the hexagon $O_1Q_3O_2Q_1O_3Q_2$ is equal to twice the area of either of the triangles; (6) the areas of the parallelograms $O_2O_3Q_2Q_3$, $O_3O_1Q_3Q_1$, $O_1O_2Q_1Q_2$ are respectively

$$2R(b+c),\quad 2R(a+c),\quad 2R(a+b).$$

Solution by W. S. F. Long, B.A., L.C.P.; *the* Proposer; *and others.*



Since O_1O_2 is the external bisector of angle C, the angle $CO_2L = \frac{1}{2}C$; similarly the angle $CO_1M = \frac{1}{2}C$, therefore $O_1Q_3 = O_2Q_3$; therefore the perpendicular from Q_3 on O_1O_2 bisects O_1O_2, and therefore passes through the centre P of the circumscribed circle of $O_1O_2O_3$. Now join O_1P, O_2P; then angle

$$O_1PO_2 = 2\angle O_3 = A + B,$$

therefore $FPO_2 = \frac{1}{2}(A+B)$, and therefore $PO_2F = \frac{1}{2}C$, therefore $PF = Q_3F$, and therefore at once $O_2Q_3 = PO_1 = 2R$; similarly O_1Q_3, O_1Q_2, &c. $= 2R$; and since O_3Q_2 is parallel to O_2Q_3, therefore $O_3Q_2Q_3O_2$ is a parallelogram. Similarly $O_1Q_2Q_1O_2$ and $O_3Q_1Q_3O_1$ are parallelograms; therefore plainly the two triangles $O_1O_2O_3$ and $Q_1Q_2Q_3$ have their sides equal, each to each, and also every two of them parallel, hence they are in perspective.

Again, since D is middle point of O_2O_3 and D_1 of Q_2Q_3, therefore $DD_1 = O_2Q_3 = 2R$, hence D_1 must be a point on the circle round ABC; hence the circle circumscribing ABC passes through the middle points of the sides of the triangle $Q_1Q_2Q_3$, and therefore for (3), &c.

From the above it is plain that the perpendiculars from the vertices of either on the sides of the other pass through the middle points of the sides of each, and therefore through the centres of circles circumscribing each.

It is evident, also, that the lines Q_1D and O_1D_1 are both equal and parallel, and therefore at once O_1Q_1 is bisected by DD_1 in O; this proves (1) finally. Also area of a parallelogram

$$O_2Q_3Q_2O_3 = O_2O_3 \cdot O_2Q_3 \sin O_3O_2Q_3$$
$$= 4R\cos\tfrac{1}{2}A \cdot 2R\sin(\tfrac{1}{2}A + C) = 2R(b+c),$$

which proves (6).

By computing the areas of the three triangles, $Q_1O_2O_3$, $Q_2O_3O_1$, $Q_3O_1O_2$, it will easily be found that their sum $= 2RS$, which proves (5).

5543. (By Robert Rawson.)—Prove that, if

$$\int \frac{d\theta}{(1+n\cos m\theta)^{\frac{1}{2}}} + \int \frac{d\phi}{(1+n\cos m\phi)^{\frac{1}{2}}} = \int \frac{d\mu}{(1+n\cos m\mu)^{\frac{1}{2}}} \quad \text{.......(1)},$$

then $\{(1+n)^{\frac{1}{2}} + (1+n\cos m\mu)^{\frac{1}{2}}\} \sin \tfrac{1}{2}(m\theta + m\phi)$

$$= \sin \tfrac{1}{2} m\mu \{(1+n\cos m\theta)^{\frac{1}{2}} + (1+n\cos m\phi)^{\frac{1}{2}}\} \quad \text{... ..(2)};$$

and, if $\displaystyle\int \frac{d\theta}{(1+n\cos m\theta)^{\frac{1}{2}}} + \int \frac{d\phi}{(1+n\cos m\phi)^{\frac{1}{2}}} = 2\int \frac{d\mu}{(1+n\cos m\mu)^{\frac{1}{2}}}$, ...(3),

then $2(1+n\cos m\mu)^{\frac{1}{2}} \sin \tfrac{1}{2}(m\theta + m\phi)$

$$= \sin m\mu \{(1+n\cos m\theta)^{\frac{1}{2}} + (1+n\cos m\phi)^{\frac{1}{2}}\} \quad \text{......(4)}.$$

I. *Solution by* J. O. Jelly, M.A.; Belle Easton; *and others.*

1. Integral (1) can be written in the form

$$\int \frac{d\theta}{(1-c^2\sin^2\theta)^{\frac{1}{2}}} + \int \frac{d\phi}{(1-c^2\sin^2\phi)^{\frac{1}{2}}} = \int \frac{d\mu}{(1-c^2\sin^2\mu)^{\frac{1}{2}}},$$

where c^2 replaces $\dfrac{2n}{1+n}$, and θ replaces $\tfrac{1}{2}(m\theta)$, &c.... ;

therefore, by a known theorem in elliptic integrals,

$$\cos\theta\cos\phi - \sin\theta\sin\phi\,(1-c^2\sin^2\mu)^{\frac{1}{2}} = \cos\mu \quad \text{...............(1)},$$

or $$\cos\theta = \cos\phi\cos\mu + \sin\phi\sin\mu\,(1-c^2\sin^2\theta)^{\frac{1}{2}} \quad \text{..............(2)},$$

or $$\cos\phi = \cos\theta\cos\mu + \sin\theta\sin\mu\,(1-c^2\sin^2\phi)^{\frac{1}{2}} \quad \text{...............(3)}.$$

Hence (2) and (3) give

$$\frac{\cos\theta - \cos\phi\cos\mu}{\sin\phi} + \frac{\cos\phi - \cos\theta\cos\mu}{\sin\theta} = \sin\mu\{(1-c^2\sin^2\theta)^{\frac{1}{2}} + (1-c^2\sin^2\phi)^{\frac{1}{2}}\}$$

or $$\sin(\theta+\phi)\left\{\frac{\cos(\theta-\phi) - \cos\mu}{\sin\theta\sin\phi}\right\} = \sin\mu\,\{(1-c^2\sin^2\theta)^{\frac{1}{2}} + (1-c^2\sin^2\phi)\}^{\frac{1}{2}}$$

$$= \sin(\theta+\phi)\,\{1 + (1-c^2\sin^2\mu)^{\frac{1}{2}}\}, \text{ by (1)}.$$

Replacing old values of c, θ, ϕ, and μ, we have

$\sin\tfrac{1}{2}(m\theta + m\phi)\,\{(1+n)^{\frac{1}{2}} + (1+n\cos m\mu)^{\frac{1}{2}}\}$

$$= \sin\tfrac{1}{2}(m\mu)\,\{(1+c\cos m\theta)^{\frac{1}{2}} + (1+c\cos m\phi)^{\frac{1}{2}}\} \quad \text{......}(a).$$

2. Let $\displaystyle\int \frac{d\theta}{(1+n\cos m\theta)^{\frac{1}{2}}} + \int \frac{d\phi}{(1+n\cos m\phi)^{\frac{1}{2}}}$

$$= 2\int \frac{d\mu}{(1+n\cos m\mu)^{\frac{1}{2}}} = \int \frac{d\sigma}{(1+n\cos m\sigma)^{\frac{1}{2}}} \text{ (say)};$$

therefore, by (a), we have

$$\frac{(1+c\cos m\theta)^{\frac{1}{2}} + (1+c\cos m\phi)^{\frac{1}{2}}}{\sin\tfrac{1}{2}(m\theta+m\phi)} = \frac{(1+n)^{\frac{1}{2}} + (1+n\cos m\sigma)^{\frac{1}{2}}}{\sin\tfrac{1}{2}(m\sigma)}$$

$$= \frac{(1+n\cos m\mu)^{\frac{1}{2}} + (1+n\cos m\mu)^{\frac{1}{2}}}{\sin\tfrac{1}{2}(m\mu+m\mu)} = \frac{2(1+n\cos m\mu)^{\frac{1}{2}}}{\sin m\mu},$$

or $\sin m\mu \left\{(1+c\cos m\theta)^{\frac{1}{2}}+(1+c\cos m\phi)^{\frac{1}{2}}\right\}$

$$= 2\sin\tfrac{1}{2}(m\theta+m\phi)\,(1+n\cos m\mu)^{\frac{1}{2}}.$$

II. *Solution by the* PROPOSER.

From (1) we obtain (considering μ as constant)

$$\frac{d\theta}{d\phi}+\frac{(1+n\cos m\theta)^{\frac{1}{2}}}{(1+n\cos m\phi)^{\frac{1}{2}}}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now, whatever form the primitive of (5) may be, one thing is clear, namely, that the value of θ in terms of n, ϕ will be of the same form as the value of ϕ in terms of n, θ.

This consideration, derived entirely from (5), suggests the following form of the primitive:—

$$(1+n\cos m\theta)^{\frac{1}{2}}+(1+n\cos m\phi)^{\frac{1}{2}}+f\left\{\tfrac{1}{2}(m\theta+m\phi)\right\}=0 \quad\ldots\ldots\ldots(6),$$

where f is an unknown function to be determined so as to satisfy (5).

Differentiating (6) in the ordinary way, we have

$$\frac{d\theta}{d\phi}+\frac{(1+n\cos m\theta)^{\frac{1}{2}}\left[f'\left\{\frac{1}{2}(m\theta+m\phi)\right\}(1+n\cos m\phi)^{\frac{1}{2}}-n\sin m\phi\right]}{(1+n\cos m\phi)^{\frac{1}{2}}\left[f'\left\{\frac{1}{2}(m\theta+m\phi)\right\}(1+n\cos m\theta)^{\frac{1}{2}}-n\sin m\theta\right]}=0\ldots\ (7).$$

Equation (7) will coincide with (5) if

$$f'\left\{\tfrac{1}{2}(m\theta+m\phi)\right\}\left\{(1+n\cos m\phi)^{\frac{1}{2}}-(1+n\cos m\theta)^{\frac{1}{2}}\right\}=n\,(\sin m\phi-\sin m\theta) \quad\ldots\ldots\ldots\ldots(8).$$

From (6), $(1+n\cos m\theta)^{\frac{1}{2}}+(1+n\cos m\phi)=-f\left\{\frac{1}{2}(m\theta+m\phi)\right\}$ ....(9).

Multiply (8) by (9), then we obtain

$$\frac{f'\left\{\frac{1}{2}(m\theta+m\phi)\right\}}{f\left\{\frac{1}{2}(m\theta+m\phi)\right\}}=-\frac{\sin m\phi-\sin m\theta}{\cos m\phi-\cos m\theta}=\frac{\cos\left\{\frac{1}{2}(m\theta+m\phi)\right\}}{\sin\left\{\frac{1}{2}(m\theta+m\phi)\right\}} \quad\ldots\ldots\ldots(10).$$

Integrating (10), we have

$$f\left\{\tfrac{1}{2}(m\theta+m\phi)\right\}=C\sin\left\{\tfrac{1}{2}(m\theta+m\phi)\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

C being an arbitrary constant. Hence the general integral, or complete primitive, of (5) is

$$(1+n\cos m\theta)^{\frac{1}{2}}+(1+n\cos m\phi)^{\frac{1}{2}}+C\sin\left\{\tfrac{1}{2}(m\theta+m\phi)\right\}=0\ldots\ldots\ldots(12).$$

In (5) and (12) take $\theta=\mu$, when $\phi=$ zero, then (1) and (2) follow.

In (5) and (12) take $\theta=\mu$, when $\phi=\mu$, then (3) and (4) follow.

5508. (By R. RAWSON.)—Prove that

$$\int_b^a \frac{dx}{\left\{(a-x)(b-x)(c-x)\right\}^{\frac{1}{2}}}=\int_{-\infty}^{c}\frac{dx}{\left\{(a-x)(b-x)(c-x)\right\}^{\frac{1}{2}}},$$

where a, b, c are in the order of magnitude.

I. *Solution by* J. HAMMOND, M.A.; L. W. JONES, B.A.; *and others.*

To reduce the first integral to a complete elliptic integral of the first kind, put $a-x=y^2$; then we have

$$\int_b^a \frac{dx}{\{(a-x)(b-x)(c-x)\}^{\frac{1}{2}}} = \int_0^{(a-b)^{\frac{1}{2}}} \frac{2dy}{\{(a-b-y^2)(a-c-y^2\}^{\frac{1}{2}}}$$

$$= \frac{2}{(a-c)^{\frac{1}{2}}} \int_0^1 \frac{dy}{\left[(1-y^2)\left(1-\frac{a-b}{a-c}y^2\right)\right]^{\frac{1}{2}}} = \frac{2}{(a-c)^{\frac{1}{2}}} F_1\left[\left(\frac{a-b}{a-c}\right)^{\frac{1}{2}}\right].$$

The second is reduced by putting $c-x=z^2$, so that

$$\int_{-\infty}^c \frac{dx}{\{(a-x)(b-x)(c-x)\}^{\frac{1}{2}}} = \int_0^\infty \frac{2dz}{\{(a-c+z^2)(b-c+z^2)\}^{\frac{1}{2}}}.$$

Again, put $z=(b-c)^{\frac{1}{2}}\tan\theta$; then the integral becomes

$$\int_0^{\frac{1}{2}\pi} \frac{2d\theta}{\{(a-c)\cos^2\theta+(b-c)\sin^2\theta\}^{\frac{1}{2}}}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{2d\theta}{\{a-c-(a-b)\sin^2\theta\}^{\frac{1}{2}}} = \frac{2}{(a-c)^{\frac{1}{2}}} F_1\left[\left(\frac{a-b}{a-c}\right)^{\frac{1}{2}}\right].$$

II. *Solution by* Professor WOLSTENHOLME, M.A.

More generally, if $a>b>c>d$, we have

$$\int_b^a \frac{dx}{\{(a-x)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}} = \int_d^c \frac{dx}{\{(a-x)(b-x)(c-x)(x-d)\}^{\frac{1}{2}}};$$

whence, putting $d=-\infty$, we get the proposed result.

Put $\frac{a-x}{x-b}=\lambda\frac{c-y}{y-d}$, so that, when $x=a$, $y=c$; and when $x=b$, $y=d$; and determine λ so that, when $x=c$, $y=a$. This gives $\lambda=\frac{a-d}{b-c}$, and we have, when $x=d$, $y=b$, since the condition for this is $\lambda=\frac{a-d}{b-c}$ also. We have then

$$\frac{a-b}{x-b}-1=\lambda\frac{c-d}{y-d}-\lambda, \text{ and } \frac{dx}{(x-b)^2}=\lambda\frac{c-d}{a-b}\frac{dy}{(y-d)^2}.$$

Also $$\frac{x-c}{x-b}=\mu\frac{y-a}{y-d}, \text{ where } \mu=-\frac{c-d}{a-b};$$

and $$\frac{x-d}{x-b}=\mu\frac{y-b}{y-d}, \text{ where } \nu=\frac{a-d}{a-b}\cdot\frac{c-d}{a-b}=-\lambda\frac{c-d}{a-b}.$$

If, then, $a > b > c > d$, as x increases from b to a, y will increase from d to c, and

$$\int_b^a \frac{dx}{\{(a-x)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}} = \int_b^a \frac{\frac{dx}{(x-b)^2}}{\left(\frac{a-x}{x-b}\cdot\frac{x-c}{x-b}\cdot\frac{x-d}{x-b}\right)^{\frac{1}{2}}}$$

$$= \int_d^c \frac{\lambda\frac{c-d}{a-b}\frac{dy}{(y-d)^2}}{\left\{\lambda\frac{c-y}{y-d}\cdot\frac{(c-d)}{a-b}\frac{a-y}{y-d}\cdot\lambda\frac{c-d}{a-b}\cdot\frac{b-y}{y-d}\right\}^{\frac{1}{2}}}$$

$$= \int_d^c \frac{dy}{\{(a-y)(b-y)(c-y)(y-d)\}} = \int_d^c \frac{dx}{\{(a-x)(b-x)(c-x)(x-d)\}^{\frac{1}{2}}}.$$

If we put $x = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\cos 2\theta$, in the original integral, we get

$$2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{[\{(a-c)\cos^2\theta+(b-c)\sin^2\theta\}\{(a-d)\cos^2\theta+(b-d)\sin^2\theta\}]^{\frac{1}{2}}},$$

or, putting $\tan\theta = x$, we obtain any one of the following expressions :

$$2\int_0^\infty \frac{dx}{[\{a-c+(b-c)x^2\}\{a-d+(b-d)x^2\}]^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots (1),$$

$$2\int_0^a \frac{dx}{[\{a-c+(a-d)x^2\}\{b-c+(b-d)x^2\}]^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots (2),$$

$$2\int_0^\infty \frac{dx}{[\{b-c+(a-c)x^2\}\{b-d+(a-d)x^2\}]^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots (3),$$

$$2\int_0^\infty \frac{dx}{[\{a-d+(a-c)x^2\}\{b-d+(b-c)x^2\}]} \quad\ldots\ldots\ldots\ldots (4),$$

$$2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\{(a-c)(a-d)-(a-b)(c-d)\cos^2\theta\}^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots (5),$$

$$2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\{(b-c)(a-d)+(a-b)(c-d)\cos^2\theta\}^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots (6);$$

whence it is obvious the integral always lies between

$$\frac{\pi}{\{(a-c)(b-d)\}^{\frac{1}{2}}} \text{ and } \frac{\pi}{\{(b-c)(a-d)\}^{\frac{1}{2}}}.$$

[Prof. WOLSTENHOLME remarks that, with respect to the corresponding values of x and y, it may be noticed that, when x is $< d$, $\frac{y-b}{y-d}$ is negative, or y lies between d and b. When $x = -\infty$, $y = \frac{ac-bd}{a+c-b-d}$, and as x increases from $-\infty$ to d, y increases from this value to b. When x lies between d and c, y lies between b and a, both increasing together; as x increases

from c to b, y first increases from a to ∞ $\left(\text{when } x = \dfrac{ac-bd}{a+c-b-d}\right)$, and then from $-\infty$ to d: as x increases from b to a, y increases from d to c, and as x increases from a to ∞, y increases from c to $\dfrac{ac-bd}{a+c-b-d}$. This value is $\quad >c$ by $\dfrac{(b-c)(c-d)}{a-b+c-d}$, and $<b$ by $\dfrac{(a-b)(b-c)}{a-b+c-d}$.
Hence x and y can never be equal to the same real quantity, or the equation $\dfrac{(a-x)(x-d)}{(x-b)(c-x)} = \dfrac{a-d}{b-c}$ has impossible roots.

This equation is

$$x^2(a-b+c-d)-2(ac-bd)x+abcd\left(\frac{1}{d}-\frac{1}{c}+\frac{1}{b}-\frac{1}{a}\right) = 0,$$

whose discriminant is

$$(ac-bd)^2+(a-b+c-d)(bcd-cda+dab-abc) = (a-b)(b-c)(c-d)(d-a),$$

or negative.]

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146. Note on Questions 5730, 5734, 5759; *by the* Editor.

These problems are solved,—as Propositions 46 (p. 174), 50 (p. 187), 27 (p. 89), respectively,—in the third edition of Mr. Whitworth's excellent little work entitled *Choice and Chance*, of which we gave a notice on p. 179 of the *Educational Times* for July last.

Questions 5734, 5759 are identical with Mr. Whitworth's propositions, Mr. Martin in Question 5734 using $a : b$ for Mr. Whitworth's μ.

Of Question 5730 Mr. Whitworth solves the first part only; but he states that "the second and third parts follow immediately" therefrom. The same question is solved by its proposer, Mr. Tebay, as follows:—

"Suppose he wins the last μ points in succession; then the preceding trial must have been a failure. This is expressed by qp^{μ}, where $q = 1-p$; and it is certain that this result was arrived at in one of $2^{n-\mu-1}$ ways (see below), all of which are not equally probable. Let $n-\mu-1 = r$, and of these points let ν be failures; then the probability that the player scores the last μ points is

$$\Sigma\left[\frac{(r)!}{(\nu)!\,(r-\nu)!}p^{r-\nu}q^{\nu}\right]qp^{\mu}, \text{ from } \nu = 0 \text{ to } r,$$

$$= \left[p^r+rp^{r-1}q+\frac{r(r-1)}{1\,.\,2}p^{r-2}q^2+\ldots+q^r\right]qp^{\mu} = qp^{\mu}.$$

Now, since μ can have all values from m to $n-1$, the sum of which is $qp^m\dfrac{1-p^{n-m}}{1-p} = p^m-p^n$, and since he can score all the points in succession,

therefore the probability that he wins with at least m points is p^m. The probability that he wins with exactly m points is qp^m. If $p = q$, these are 2^{-m} and 2^{-m-1}. Again, suppose the player finally scores μ points accidentally; then the $(n-\mu)^{\text{th}}$ trial must have been a failure. Of the remaining $n-\mu-1$ points, suppose ν are failures; then the number of ways in which he can finally score μ points is

$$\Sigma \frac{(n-\mu-1)\,!}{(\nu)\,!\,(n-\mu-1-\nu)\,!}, \text{ from } \nu = 0 \text{ to } n-\mu-1, \;= 2^{n-\mu-1}.$$

"Since μ can have all values from m to $n-1$, the number of ways in which the player can finally score m points $= \Sigma \,.\, 2^{n-m}+1 = 2^{n-m}$.

"The whole number of ways in which the event can happen and fail is

$$\Sigma \frac{(n)\,!}{(\nu)\,!\,(n-\nu)\,!} \text{ from } \nu = 0 \text{ to } n, \;= 2^n.$$

"Hence the probability that the player wins with at least m points is 2^{-m}. The probability that the player wins with exactly m points is $2^{-(m+1)}$, the same as before. These results may be tested by particular examples. If $n = 5$, and 1 denotes success, and 0 failure, we have the following arrangement:—

00001	01101	00011	00111	11110	10010	11100	11000
00101	10101	01011	10111	11010	01010	10100	01000
01001	11001	10011	01111	10110	00110	01100	10000
10001	11101	11011	11111	01110	00010	00100	00000

Thus the probability that he wins with at least 2 points $= \frac{8}{32} = \frac{1}{4}$, and with exactly 2 points $\frac{4}{32} = \frac{1}{8}$. It is remarkable that the above results are independent of n.

"In my solution of Question 5564 [see *Reprint*, Vol. XXIX., pp. 72, 73], I have calculated the sum of all the points which can be finally scored by the player, and also the sum of all the points which are forfeited. I have deviated from this method in the above investigation, as it only seems necessary to consider the number of ways in which the events can happen. I would here notice an oversight in the latter part of the solution referred to above. I have there stated, 'he can finally fail as often as he can finally succeed, namely, 2^n-1 times.' This implies that all the previous failures are forfeited. To rectify this oversight, the sum of all the failures is

$$\Sigma \left[\nu \frac{(n)\,!}{(\nu)\,!\,(n-\nu)\,!}\right] \text{ from } \nu = 0 \text{ to } n,$$

$$= n + 2\frac{n(n-1)}{1\,.\,2} + \ldots + n = n\left\{1 + \frac{n-1}{1} + \frac{(n-1)(n-2)}{1\,.\,2} + \ldots + 1\right\} = n\,.\,2^{n-1}.$$

Therefore the respective probabilities are

$$\frac{2^n-1}{(n+2)\,2^{n-1}-1}, \quad \frac{n\,.\,2^{n-1}}{(n+2)\,2^{n-1}-1}."$$

Mr. Tebay goes on to state that he has herein given a part only of Question 5730, inasmuch as, he adds, "the other cases, though neat in themselves, present almost insurmountable difficulties which I had not foreseen;"—a remark which is in singular contrast to Mr. Whitworth's "follow immediately," cited above.

147. Reply to Professor Monck's Note on Question 5502.
By W. S. B. Woolhouse, F.R.A.S.

In Volume XXIX of the *Reprint*, page 62, I have for the first time seen this *Note*, which begins as follows:—"The chance that a random point P will be in the arc α is undoubtedly $\frac{\alpha}{\alpha+\beta+\gamma+\delta}$, and the chance that a second random point P′ will also lie in the same arc being identical, the chance that the chord PP′ will lie in it is $\frac{\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$." As Professor Monck here expressly counts PP′ as a distinct chord from P′P, I have to observe that the last mentioned chance is $\frac{\frac{1}{2}\alpha^2}{(\alpha+\beta+\gamma+\delta)^2}$, and furthermore I have to add that every subsequent statement in the Note is erroneous.

Again, the editorial note on page 59 gives a statement of Professor Monck's containing the following:—"The whole dispute seems to me to turn on whether certain chords ought to be counted twice over or once only. Something may be said for Mr. Woolhouse's way of counting (which is really counting the chord *once* when both extremities are in the same arc, and *twice* when they are in different arcs), but he has not said it." It is as well that I have not said it, for the statement is again incorrect. In my solution every chord is counted twice; and it is essential to accuracy that the counting should be precisely the same throughout.

Professor Monck proceeds thus: "Suppose, for example, the question were, What is the chance that a random chord will intersect a fixed diameter, is it not clearly $\frac{1}{2}$? Yet this can only be reached, I apprehend, by counting the chords which intersect the diameter twice, and those which do not intersect it once only." I assert, on the contrary, that it can only be reached by counting all chords alike, otherwise the value, which if correct, would not be obtained.

5653. (By E. W. Symons.)—Find the radii and direction-cosines of the central circular sections of the conicoid

$$\phi(x, y, z) \equiv a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy - 1 = 0,$$

and apply the result to prove that, if $\begin{vmatrix} l, & m, & n \\ x, & y, & z \\ \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \end{vmatrix}$ vanish identically for all values of x, y, z which simultaneously satisfy the equations

$$lx + my + nz = 0, \qquad \phi(x, y, z) = 0,$$

then will $$4a^2b^2c^2r^6 - r^2(a^2+b^2+c^2) + 1 = 0,$$

and $$\frac{m^2+n^2}{(bn+cm)^2} = \frac{n^2+l^2}{(cl+am)^2} = \frac{l^2+m^2}{(am+bl)^2} = r^2,$$

where $$r^2 \equiv x^2+y^2+z^2.$$

Solution by the PROPOSER; W. J. C. SHARP, B.A.; *and others.*

The equation $x^2+y^2+z^2=r^2$ is a sphere concentric with $\phi(x,y,z)=0$, and $\quad (1-a^2r^2)x^2+\ldots\ldots+\ldots\ldots+2bcr^2yz+\ldots\ldots+\ldots\ldots=0 \ldots\ldots\ldots\ldots(1)$ is a cone with centre for vertex and passing through the curve of intersection of the sphere and conicoid. If this cone degenerate into two planes, they will be central circular sections. The condition for this is

$$(1-a^2r^2)(1-b^2r^2)(1-c^2r^2)+2a^2b^2c^2r^6-(1-a^2r^2)b^2c^2r^4 \\ -(1-b^2r^2)c^2a^2r^4-(1-c^2r^2)a^2b^2r^4=0,$$

or
$$4a^2b^2c^2r^6-r^2(a^2+b^2+c^2)+1=0,$$

which gives the radii of the central circular sections. Again, (1) must be identical with $\quad (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)=0,$

$$\therefore\ \frac{l_1l_2}{1-a^2r^2}=\frac{m_1m_2}{1-b^2r^2}=\frac{n_1n_2}{1-c^2r^2}=\frac{m_1n_2+m_2n_1}{2bcr^2}=\frac{n_1l_2+n_2l_1}{2car^2}=\frac{l_1m_2+l_2m_1}{2abr^2};$$

therefore $\quad \dfrac{m_1}{n_1}\cdot\dfrac{m_2}{n_2}=\dfrac{1-b^2r^2}{1-c^2r^2},\qquad \dfrac{m_1}{n_1}+\dfrac{m_2}{n_2}=\dfrac{2bcr^2}{1-c^2r^2};$

therefore $\dfrac{m_1}{n_1},\ \dfrac{m_2}{n_2}$ are the roots of

$$\frac{m^2}{n^2}-\frac{2bcr^2}{1-c^2r^2}\cdot\frac{m}{n}+\frac{1-b^2r^2}{1-c^2r^2}=0;$$

therefore
$$m^2+n^2=r^2(bm+cn)^2;$$

and, by symmetry, $\quad \dfrac{m^2+n^2}{(bn+cm)^2}=\dfrac{n^2+l^2}{(cl+an)^2}=\dfrac{l^2+m^2}{(am+bl)^2}=r^2$

are equations giving the direction-cosines.

For the second part it is sufficient to observe that, if the normals at all points of a plane section of a conicoid pass through the centre of the section, then the section must be a circle. Now the vanishing of the above determinant expresses this condition, which is necessary and sufficient; and therefore all results arrived at by the first method of treating the problem must hold for the second method; and therefore &c.

5723. (By the Rev. W. A. WHITWORTH, M.A.)—If a coin be tossed n times, prove that the chance that there are not two consecutive heads is

$$\frac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{4^{n+1}\sqrt{5}}.$$

Solution by J. HAMMOND, M.A.; FR. WERTSCH; *and others.*

Let $u_n=$ the chance of not throwing two consecutive heads in n throws; then $u_n=$ chance of tail in first toss $\times\, u_{n-1}$ + chance of head in first toss and tail in second $\times\, u_{n-2}$; that is, $\quad u_n=\frac{1}{2}u_{n-1}+\frac{1}{4}u_{n-2}.$

The solution of this equation is $\quad u_n=\mathrm{A}\alpha^n+\mathrm{B}\beta^n,$

where α,β are the roots of the equation $\quad x^2-\frac{1}{2}x-\frac{1}{4}=0;$

therefore $$u_n = A\left(\frac{1+\sqrt{5}}{4}\right)^n + B\left(\frac{1-\sqrt{5}}{4}\right)^n.$$

The constants are determined from the conditions $u_0=1$, $u_1=1$; hence

$$A+B=1,\quad A\left(\frac{1+\sqrt{5}}{4}\right)+B\left(\frac{1-\sqrt{5}}{4}\right)=1.$$

Solving these, we obtain $A = \dfrac{(1+\sqrt{5})^2}{4\sqrt{5}}$, $B = -\dfrac{(1-\sqrt{5})^2}{4\sqrt{5}}$;

thus, $$u_n = \frac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{4^{n+1}\sqrt{5}}.$$

[Mr. WHITWORTH remarks that, more generally, the chance that there shall not be κ consecutive heads is $2^{-n}(a_{n+1}-a_n)$, where a_r is the coefficient of x^r in the expansion of $(1-2x+x^{\kappa+1})^{-1}$.]

5735. (By H. L. ORCHARD, B.A., L.C.P.)—If ABC be a plane triangle in which $A=90°$, $B=60°$, $C=30°$, and if PQR be the triangle formed by joining the centres of the escribed circles; prove that

$$\Delta PQR = 2(\sqrt{3}+1)\,\Delta ABC.$$

Solution by Prof. COCHEZ; J. O'REGAN; H. MURPHY; *and others.*

TODHUNTER a démontré (*Plane Trigonometry*, page 198) que l'on a

$$\Delta PQR = \Delta ABC\left(1+\frac{a}{b+c-a}+\frac{b}{a+c-b}+\frac{c}{a+b-c}\right).$$

Or dans le cas actuel si on désigne le côté opposé à l'angle de 90°, on a

$$b = a\sin 30° = \tfrac{1}{2}a,\quad c = a\sin 60° = \tfrac{1}{2}a\sqrt{3}.$$

Des lors, en remplaçant et réduisant, il vient

$$\Delta PQR = \Delta ABC\left(1+\frac{2}{\sqrt{3}-1}+\frac{1}{1+\sqrt{3}}-\frac{\sqrt{3}}{3-\sqrt{3}}\right) = 2(\sqrt{3}+1)\Delta ABC.$$

5576. (By J. C. MALET, M.A.)—Prove that (1) the centres of the osculating circles of an equilateral hyperbola, which pass through the centre of the curve, lie on the curve; and (2) the inflexional tangents of the Lemniscate $r^2 = a^2\cos 2\theta$ touch the hyperbola $4r^2\cos 2\theta = a^2$.

Solution by the PROPOSER.

1. The equation of the hyperbola being $x^2-y^2=a^2$, the equations of the required circles are easily found to be included in the form

$$x^2+y^2\pm ax\sqrt{2}\pm ay\sqrt{(-2)}=0,$$

the radius of each of these four circles is equal to 0, which proves (1).

2. Inverting, we find for the inflexional tangents, distinct from the nodal tangents, of the Lemniscate $r^2 = a^2 \cos 2\theta$, the equations

$$x \pm y\sqrt{(-1)} \pm \frac{a}{\sqrt{(2)}} = 0,$$

which together with the nodal tangents envelop the hyperbola

$$4r^2 \cos 2\theta = a^2.$$

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148. Remarks on Mr. Artemas Martin's Note respecting Solutions to Certain Questions on Probability.*

By W. S. B. Woolhouse, F.R.A.S.

Having on former occasions made definite statements on the point raised by Mr. Martin, it will not be requisite on my part to say anything on that head. In addition to Mr. Martin's quotation from the *Reprint*, Vol. XXVII, page 79, I may refer to an analogous statement in Vol. VII, page 82, from which the following is taken :—

" For the accurate investigation of all questions involving direct values of probability, two conditions are absolutely essential to the usual process of calculation.

" 1. All possible ways, both for and against the specified event, should individually possess an equal degree of probability.

" 2. In estimating relatively the number of ways which fulfil the proposed event, and the total number of ways, it is indispensable that all the equally probable elements shall be alike included, without exception, or severally repeated an equal number of times.

" When these two conditions subsist, it is evident that the required value of the sought probability is correctly expressed by the fraction

$$\frac{\text{Number of ways fulfilling the event}}{\text{Total number of ways}}.$$

If, however, either of the two conditions be wanting, it is equally evident that the reasoning entirely fails, and, therefore, that the truth of the result cannot in such case be relied upon."

Now, referring to the three questions adduced by Mr. Martin as examples, I have to state that the solutions given to the first and second questions, and Mr. Heaton's solution to the third question are correct, and strictly in accordance with the principles above stated.

In the solution of the question No. 1, the points of division, C and D, are random points in the line AB. As only one trial is supposed to be made as regards the placing of the three pieces, it follows that, if the operations stated in the question were repeated indefinitely, only one trial would be associated with each division into three parts, so that the total number of trials is really the number of ways in which the plank is

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* See pp. 46—48 of this volume.

capable of being divided, the same being represented by $\iint dy\,dx$; also these ways are considered as equally probable, because no reason is assignable why one should be more probable than another.

The enunciation of the second question is general and comprehensive; we are not informed of any process by which the cone and plane are brought into existence. We are therefore bound to take into consideration every possible configuration of diagram. To make the solution practicable, it is further expressly assumed that the generating line of the cone is constant, and that all values of the angle at the vertex are equally probable.

According to the third question, part of the water contained in the second cask is poured into the first, and then part of the mixture is poured back into the second. If the compound operation were repeated an indefinite number of times, since after each first operation the second is understood to have only one trial, it follows, as before, that the total number of trials is simply the number of ways of performing the first operation alone, which is represented by $\int dx$ or b; and all these ways are accounted equally probable, because there is no assignable reason to the contrary, which, indeed, is tacitly assumed in the question. The alternative solution, which Mr. Martin has offered as a mere suggestion for the purpose of enquiry, does not apply the formula in accordance with the conditions of the question, and is therefore not correct.

5737. (By S. A. Renshaw.)—If from the vertices of a quadrilateral inscribed in a circle perpendiculars be drawn to the sides and diagonals, prove that the four lines passing through the feet of the perpendiculars, taken in sets of three, as drawn from each vertex, all pass through the same point.

I. *Solution by* F. D. Thomson, M.A.; Simonelli Ruggero; *and others.*

Let ABCD (Figures 1 and 2) be the quadrilateral inscribed in the circle centre O. Draw DX, AZ, ON perpendicular to BC, DY perpendicular to AB, and AT perpendicular to DC.

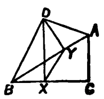


Fig. 1.

Then YXC = BDY = complement of DBA. Similarly TZB = CAT = complement of ACD; but DBA = ACD, therefore YXC = TZB, and therefore the two lines XY, ZT make with BC an isosceles triangle, QXZ suppose (Fig. 2), of which the vertical angle is equal to DOA.

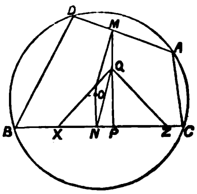


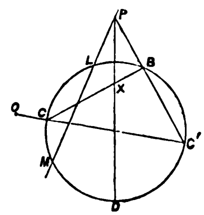
Fig. 2.

Draw QP perpendicular to BC; this will bisect XZ, and, if produced to meet DA in M, it will also bisect DA. Join OM, QN. Then, by similar triangles, $\frac{QP}{OM} = \frac{PX}{DM} = \cos\theta$, if θ is the angle between DA and BC; also $\frac{NP}{OM} = \sin\theta$, therefore

$NP^2 + QP^2 = OM^2$, therefore $OM = NQ$, and therefore OM is parallel to QN. Join OQ, MN, these will bisect each other in G, and $OQ = 2OG$.

Now G is the centroid of the four points A, B, C, D, therefore Q is a point symmetrical with respect to the four points; which proves the theorem.

II. *Solution by* F. D. THOMSON, M.A.; Prof. JOHNSON, M.A.; *and others*.



1. Let O be any point, and LM its pole with respect to a conic; and let CC′ be a chord passing through O, and A, B, D other points on the curve. Let BC′ meet LM in P, and join DP, meeting BC in X; and let Y and Z be points obtained in a similar manner from CA and AB. Then X, Y, Z lie on a straight line. This is proved, as in *Reprint*, Vol. III, p. 39, by taking the point A as given by $L : R : M = 1 : a : a^2$, and similarly for the other points; L, M being the tangents through O, and R the polar of O.

It is shewn that the equation of XY is

$$2dabcL + \{d^3 + d^2(a+b+c) - d(ab+bc+ca) - abc\} R - 2d^2M = 0 \ldots (1),$$

which is symmetrical in a, b, c, and must therefore be the equation to the line XYZ.

2. If we draw the corresponding lines from A on the sides of the triangle BCD, we shall get, in the same way, the line

$$2abcdL + \{a^3 + a^2(b+c+d) - a(bc+ca+ad) - bcd\} R - 2a^2M = 0 \ldots (2).$$

If $\qquad a+b+c+d = \Sigma a,$ and $abc+bcd+cda+dab = \Sigma abc,$

equation (1) may be written $2abcdL + (d^2\Sigma a - \Sigma abc) R - 2d^2M = 0,$

and (2) may be written $2abcdL + (a^2\Sigma a - \Sigma abc) R - 2a^2M = 0$;

and therefore (1) and (2) meet in the point given by

$$\frac{L}{\Sigma abc} = \frac{R}{2abcd} = \frac{M}{abcd\,\Sigma a} \ldots\ldots\ldots\ldots (3),$$

which is symmetrical in A, B, C, D. Hence all the four lines meet in the point given by (3).

When LM is the line at infinity, and the conic is a circle, the proposition becomes the one stated in the question.

3. These results may easily be transformed to corresponding ones for Cartesian coordinates. For $L : R : M = x+iy : r : x-iy$, where r is the radius of the circle, and $i^2 = 1$.

Thus the point A is given by $\dfrac{x+iy}{1} = \dfrac{r}{a} = \dfrac{x-iy}{a^2}$, and so on; hence

$$a = \frac{x-iy}{r}, \quad \Sigma a = \frac{1}{r}\{\Sigma x - i\Sigma y\}, \quad \Sigma\frac{1}{a} = \frac{1}{r}\{\Sigma x + i\Sigma y\}.$$

Hence (3) becomes $\quad \dfrac{x+iy}{\Sigma x + i\Sigma y} = \dfrac{r}{2r} = \dfrac{x-iy}{\Sigma x - i\Sigma y} = \dfrac{x}{\Sigma x}$;

therefore $\qquad x = \frac{1}{2}\Sigma x, \quad y = \frac{1}{2}\Sigma y.$

Hence the point is obtained by doubling the vector from the centre to the centroid of the four points.

III. *Solution by the* PROPOSER.

Constructing the figure, it is evident that the angles ZSQ, SQZ, ZKI, KIZ, DBA are all equal, and ZQK = QKZ; therefore ZS = ZQ = ZK = ZI; that is, QI, SK bisect one another in Z.

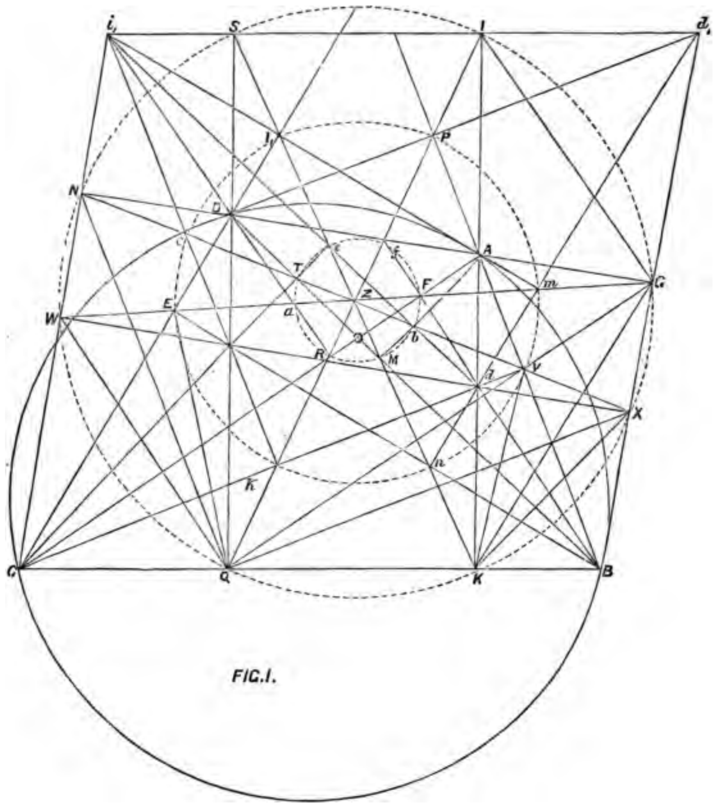


FIG. I.

Again, a circle passes through the points Q, E, P, G, B, D; therefore ∠ PQG = ABG = EBC = FGQ, wherefore ZQ = ZG. Also ∠ QEZ = QDG = NCQ; therefore a circle passes through the points C, W, Q, E; therefore ∠ QWZ = BCD. Also the angle WQZ is composed of the angles WQE, EQD, DQZ, which are plainly equal respectively to the angles WCE, EBD, DCA; that is, to the angles VCB, ACV, DCA, the sum of which is the angle BCD; wherefore the ∠ QWZ = WQZ and ZQ = ZW.

Again, ∠ GAB = GKB, since the points G, A, K, B are in a circle; and, for the same reason, ∠ GDB = GQB, therefore ∠ DBA = QGK; but DBA = QIK, therefore QIK = QGK, and the points I, G, Q, K are in a

circle. But the angle QGI is a right angle (since it is equal to the angle QKI), and ZQG has been shown to be equal to ZGQ; therefore Z is the centre of the circle IGQK, and ZQ = ZI; wherefore WG, IQ bisect one another in Z. Similarly, NX and SK may be shown to bisect one another; therefore the four lines QI, SK, WG, NX mutually bisect each other, and therefore pass through the same point (Z). It is evident, by symmetry, that ZL = Z*c* = ZE = Z*k* = Z*n* = ZV = Z*m* = ZP. It may also be shown, after what has been proved, that Z is in like manner the centre of a circle passing through *e*, T, *a*, R, M, *b*, F, *f*, for the angle ZMR = LAC = ZKC = ZQK = PDB = PRM; and, since the angles *e*TM, *f*FR are right, it follows that Z*e* = ZT = Z*a* = ZR = ZM = Z*b* = ZF = Z*f*.

2. If, from any two of the vertices of an inscribed quadrilateral, perpendiculars be drawn to the opposite side, or one of the diagonals, and from either end of that side or diagonal a third perpendicular be drawn to the side or diagonal joining the two vertices taken, the segment of the line passing through the feet of the three perpendiculars drawn from either of the first taken vertices, upon the sides of the triangle formed by the other three, and included between the two perpendiculars first drawn, will subtend a right angle at the foot of the third perpendicular. As examples of this property see the angles QGI, LVN, *a*MF, where the middle letter indicates the right angle.

In the above solution the values of the angles of three of the isosceles triangles, whose common vertex is Z, is given, and it may be also observed that since $\angle$ ZGQ = ABG and $\angle$ QGK = DBA, therefore ZGK = DBG. Also, since $\angle$ ZQG = GBA and VQX = FBA, therefore ZQX = FBG = DAC.

If perpendiculars be drawn from the ends of the common side of any two of the triangles that compose the inscribed quadrilateral, to the side or diagonal joining the remaining vertices, the Simson lines passing through the feet of these perpendiculars, meet the same (alternately) in two points such that the line joining them passes through the orthocentres of the triangles chosen.

To prove this, let d be the orthocentre of the triangle ABC, and join dX; then a circle passes through the points d, K, B, X, V; therefore $\angle d$XV = dBV = GNX; therefore dX is parallel to AD, and consequently coincides with WX. Wi (i being the orthocentre of the triangle BCD) will, in like manner, coincide with WX. Wherefore WX passes through d and i.

Again, $\angle c$DN = PDA = ABG = EBC, and DNc = DBE; therefore DcT = DBC = DAC, and the angles at R and N are right; therefore $\angle$ Ci_1D = DAC = DcT; therefore a circle passes through c, N, i_1, D, L, and i_1cD is a right angle; therefore i_1c is parallel to AB, and therefore coincides with ck. Also, since the $\angle$ DcT = DBC = DiT; therefore a circle passes through D, c, E, i, T, and Dci is a right angle, and ci is parallel to AB, and coincides with ck; ck therefore passes through i and i_1. Or, again, since N, C, T, D are in a circle, the $\angle$ NDC = NTC; but NDC = Ni_1A, and NTC = NbM; therefore N, i_1, b, A are in a circle; therefore $\angle$ Nbi_1 = NAi = 90° − ABC = VCB = VTB (since V, T, C, B are in a circle); therefore bi_1 is parallel to DB, and therefore coincides with eb.

Otherwise: $\angle$ LbN = LAD = LMD = bTM (since ZT = ZM); therefore bi_1 is parallel to DB, and coincides with eb.

Also, since $\angle$ VTB = VCB, therefore $\angle$ VbA = ABC = AdV; therefore b, d, V, A are in a circle, and the $\angle db$V = dAV = VCB = VTB; therefore bd is parallel to DB, and coincides with eb, wherefore eb passes through d and i_1.

3. **The orthocentres of the four triangles, composing the quadrilateral, lie on a circle.**

To prove this, join NQ, KG; then, since LN = ZQ and $Zc = Zk$; therefore ii_1 is parallel to NQ, similarly dd_1 is parallel to KG. Hence $Lii_1N = QNC = QDC = LDS = Li_1S$; therefore $\angle Si_1i = Li_1N = CDN = ABC = AGK = d_1dX$; therefore the orthocentres i_1, i, d, d_1 are in a circle.

4. The reciprocal of the property (respecting the orthocentres) may easily be proved for any conic by means of the generating circle, and a figure of the ellipse is added (Fig. 2.) in illustration. In regard to this figure,

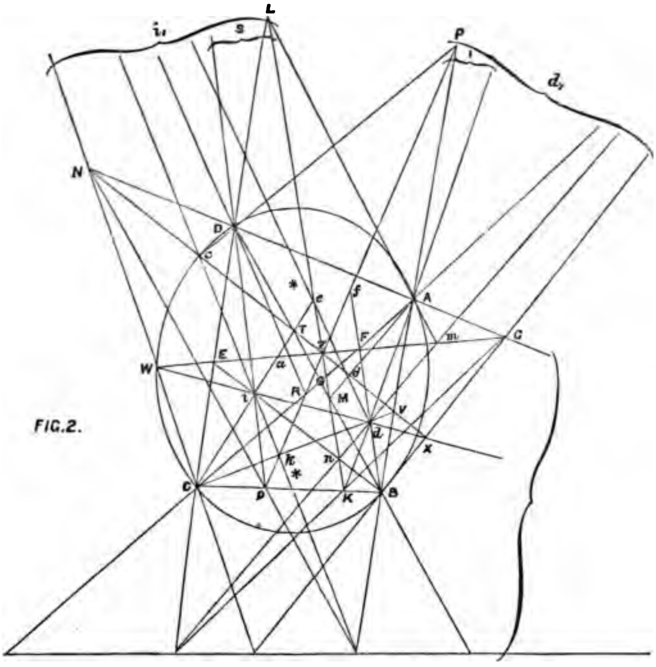


FIG. 2.

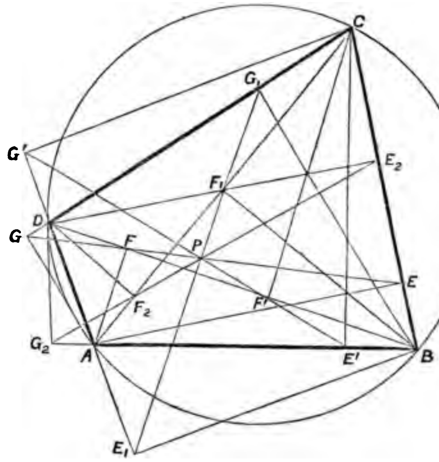
is sufficient to remark that the analogues of those lines, which in the circle are at right angles, do in the conic include between them intercepts on the directrix that subtend right angles at the focus (or *vice versâ*, if centre be read for focus, &c.); and the anologues of those that are parallel in one curve, in the other meet in the same point on the directrix.

5. If the point A revolve in the circle whilst B, C, D remain fixed, the line i_1d_1 will move constantly parallel to CB.

6. The inscriptibility of the orthocentres may be otherwise proved from the consideration that each of the figures i_1iBA, d_1dCD is a parallelogram, and therefore $i_1i = AB$ and $dd_1 = DC$; and plainly $id = AD$ and $i_1d_1 = CB$, wherefore the figure i_1idd_1 is in all respects equal and similar to the original figure ABCD, and therefore inscriptible.

IV. *Solution by* E. RUTTER; T. MORLEY, L.C.P.; *and others.*

Let E, F, G be the feet of the perpendiculars from A on the sides and diagonal; E′, F′, G′ those from C; E_1, F_1, G_1 those from B; and E_2, F_2, G_2



those from D; then AG_2GDFF_2, AFF_1EBE_1, $BE'F'F_1G_1C$, $CG_1DF_2F'E_2$, $CG'GAE_1E$, $DG_2E_1BE_2G_1$ are circles on the diameters AD, AB, BC, CD, AC, BD respectively; and because (AE, DE_2), (BE_1, CG) are parallels and perpendiculars on BC, AD, it is obvious that $G'E_2EE_1$, $GG_2E_1G_1$ are circles. Suppose G_2E_2, GE intersect at P; then $PE_2 = PE$, $PG = PG_2$, $PF = PF_2$, $PF_1 = PF'$.

Now, wherever GF, E_1F' intersect, they make the $\angle BFE = DF_1G'$, because $\angle DAG = DCG' = DFG = DF_1G = BFE = BF_1E'$. Similarly the intersection of G_1F_1 and G_2F_2 makes $\angle AF_2G_2 = AF_1E_1 = CF_1G_1 = E_2F_2F_1$; hence those lines all intersect in the point P.

It is evident that P is the centre of the circles $G'E_2EF_1$, $GG_2G'E'$, $FF_1F'F_2$; and that the quadrilaterals $CG'DE_2$, $CG'AE$, BE_1AE, BE_1DE_2 are inscriptible in circles whose centres are in a straight line.

5749. (By J. L. McKENZIE, B.A.)—Find the equation of the locus of principal centres of curvature, or "surface of centres," of a given central quadric surface; and prove that, if a, b, c be the semi-axes of the given quadric, and α, β, γ those of any confocal quadric, then the envelope of another concentric and coaxial quadric, with semi-axes $\frac{\alpha^2}{a}$, $\frac{\beta^2}{b}$, $\frac{\gamma^2}{c}$, is the surface of centres of the given quadric.

I. *Solution by* Professor TOWNSEND, F.R.S.

Since, for any point xyz on any central quadric abc, the two centres of curvature $x_1y_1z_1$ and $x_2y_2z_2$ are the two poles of the tangent plane with respect to the two confocal quadrics $a_1b_1c_1$ and $a_2b_2c_2$ through the point, we have consequently

$$\frac{x}{a^2}=\frac{x_1}{a_1^2}=\frac{x_2}{a_2^2},\quad \frac{y}{b^2}=\frac{y_1}{b_1^2}=\frac{y_2}{b_2^2},\quad \frac{z}{c^2}=\frac{z_1}{c_1^2}=\frac{z_2}{c_2^2}.$$

from which, since from the equation of the original surface

$$\frac{a^2x_1^2}{a_1^4}+\frac{b^2y_1^2}{b_1^4}+\frac{c^2z_1^2}{c_1^4}=1,\qquad \frac{a^2x_2^2}{a_2^4}+\frac{b^2y_2^2}{b_2^4}+\frac{c^2z_2^2}{c_2^4}=1,$$

and from those of the two confocal surfaces,

$$\frac{a^2xx_1}{a_1^4}+\frac{b^2yy_1}{b_1^4}+\frac{c^2zz_1}{c_1^4}=1,\qquad \frac{a^2xx_2}{a_2^4}+\frac{b^2yy_2}{b_2^4}+\frac{c^2zz_2}{c_2^4}=1,$$

it appears at once that the normal at xyz to the original quadric abc is a tangent at $x_1y_1z_1$ and $x_2y_2z_2$ to the two derived quadrics $\frac{a_1^2}{a}\,\frac{b_1^2}{b}\,\frac{c_1^2}{c}$ and $\frac{a_2^2}{a}\,\frac{b_2^2}{b}\,\frac{c_2^2}{c}$, and in consequence that the surface of centres of the former is the envelope, to the parameter λ, of the system of quadrics

$$\frac{a^2x^2}{(a^2+\lambda^2)^2}+\frac{b^2y^2}{(b^2+\lambda^2)^2}+\frac{c^2z^2}{(c^2+\lambda^2)^2}=1,$$

which is the second part of the question.

As regards the first part, the actual equation, in rectangular coordinates, of the surface of centres of the ellipsoid, which, from the high degree of the surface, is of some length and complexity, was first obtained by Dr. SALMON, and made the subject of a communication to the British Association, then meeting in Dublin, in the year 1857; and is given, with various processes for its investigation, in the several editions of his *Analytical Geometry of Three Dimensions*, all of which have come out since that year.

II. *Solution by* R. F. SCOTT, M.A.; E. WRIGHT, B.A.; *and others.*

If X, Y, Z, be any point, the equations giving the feet of the normals from this point to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\text{are}\quad \frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}}=\frac{Z-z}{\frac{z}{c^2}}=\theta\ldots\ldots(1);$$

therefore $$x=\frac{a^2X}{a^2+\theta},\ y=\frac{b^2Y}{b^2+\theta},\ z=\frac{c^2Z}{c^2+\theta}\ \ldots\ldots\ldots\ldots(2);$$

hence $$\frac{a^2X^2}{(a^2+\theta)^2}+\frac{b^2Y^2}{(b^2+\theta)^2}+\frac{c^2Z^2}{(c^2+\theta)^2}=1\ \ldots\ldots\ldots\ldots(3).$$

This equation gives six values for θ; and thence, from (2), we get the feet of the normals.

If two of the normals coincide, the point (X, Y, Z) is a point on the surface of centres. The condition for this is that (3) should have two equal roots;

therefore $$\frac{a^2X^2}{(a^2+\theta)^3}+\frac{b^2Y^2}{(b^2+\theta)^3}+\frac{c^2Z^2}{(c^2+\theta)^3}=0\ \ldots\ldots\ldots\ldots(4).$$

Eliminating θ between (3) and (4), we shall get the surface of centres. Or if we write $\alpha^2 = a^2+\theta$, $\beta^2 = b^2+\theta$, $\gamma^2 = c^2+\theta$, we get Mr. McKenzie's result, viz., the surface of centres is the envelope of a quadric with semi-axes $\dfrac{\alpha^2}{a}$, $\dfrac{\beta^2}{b}$, $\dfrac{\gamma^2}{c}$, α, β, γ being the axes of a confocal to the given ellipsoid.

We may now readily obtain the equation to the parallel surface of the ellipsoid. Clearly, $\theta = pr$, where p is the perpendicular from the centre on the tangent plane at (x, y, z), and r is the distance between the points (x, y, z), (X, Y, Z). For the parallel surface, $r = c$.

Thus, $$\frac{c^2}{\theta^2} = \frac{1}{p^2} = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} = \frac{X^2}{(a^2+\theta)^2}+\frac{Y^2}{(b^2+\theta)^2}+\frac{Z^2}{(c^2+\theta)^2} \quad \ldots\ldots(5).$$

And the parallel surface is got by eliminating θ between (3) and (5).

Multiplying (5) by θ, and adding to (3),
$$1+\frac{c^2}{\theta} = \frac{X^2}{a^2+\theta}+\frac{Y^2}{b^2+\theta}+\frac{Z^2}{c^2+\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The parallel is got by the elimination of θ between (5) and (6), which is Dr. Salmon's result.

For the paraboloid $$\frac{x^2}{a}+\frac{y^2}{b} = 2Z,$$

the equation for the normals is $$\frac{aX^2}{(a+\theta)^2}+\frac{bY^2}{(b+\theta)^2} = Z+\theta \quad \ldots\ldots\ldots\ldots(\text{i.})$$

Writing the equations
$$\frac{2aX^2}{(a+\theta)^3}+\frac{2bY^2}{(b+\theta)^3}+1 = 0, \quad \frac{X^2}{(a+\theta)^2}+\frac{Y^2}{(b+\theta)^2}+1 = \frac{c^2}{\theta^2} \ldots\ldots(\text{ii., iii.}),$$

we can show that the surface of centres of the paraboloid is got by eliminating θ between (i.) and (ii.), and the parallel surface by eliminating θ between (i.) and (iii.)

5761. (By F. C. Wace, M.A.)—When in a triangle ABC, there are given A, b, a, and there are two triangles that satisfy the conditions, prove that—(1) these triangles have equal circumscribing circles; and (2) if r_1, r_2 be the radii of their inscribed circles, then
$$(r_1 \sim r_2)(a+b)\tan B = 2(r_1 - a\sin B)(r_2 - a\sin B).$$

Solution by T. Mitcheson, B.A., L.C.P.; A. Buchheim; *and others.*

1. If R_1, R_2, be the radii of the circumscribing circles, we have
$$R_1 = R_2 = \tfrac{1}{2}a \operatorname{cosec} A.$$

2. If c, c' be the third sides of the triangles, we have
$$r_1 = \frac{ac\sin B}{a+b+c}, \quad r_2 = \frac{ac'\sin B}{a+b+c'},$$

therefore $\quad r_1 \sim r_2 = \dfrac{a \sin B\,(a+b)(c \sim c')}{(a+b+c)(a+b+c')}$, and $c \sim c' = 2a \cos B$;

therefore $\quad (r_1 \sim r_2)\,(a+b)\tan B = \dfrac{2a^2 \sin^2 B\,(a+b)^2}{(a+b+c)\,(a+b+c')}$

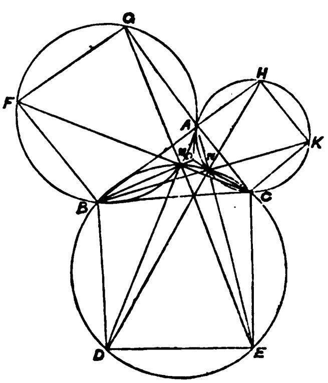
$$= 2a^2 \sin^2 B \left(\frac{c}{a+b+c} - 1\right)\left(\frac{c'}{a+b+c'} - 1\right)$$

$$= 2\left(\frac{ac \sin B}{a+b+c} - a \sin B\right)\left(\frac{ac' \sin B}{a+b+c'} - a \sin B\right)$$

$$= 2\,(r_1 - a \sin B)\,(r_2 - a \sin B).$$

5757. (By R. E. Riley, B.A.)—In the Figure for Euclid I. 47, if AD, FC meet at M; AE, BK at N; and BM, CN at O; prove that O is the radical centre of the circles that circumscribe the three squares in the figure.

Solution by H. Murphy; E. Rutter; J. F. Wilson; *and others.*



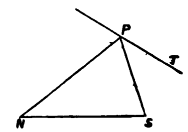
Because FC, AD meet at right angles, the points M, B, C are on the circumference of the circle circumscribing the square on BC. Similarly, the circle circumscribing the square on AB passes through M; hence BM is the radical axis of these two circles. In the same way, CN is the radical axis of the other two circles; hence O is the radical centre of the three circles. The line EG passes through M, and DH through N, and OA bisects the angle BAC.

5714. (By Professor Minchin, M.A.)—Given the base NS of a triangle NPS, and also the sum of the cosines of the base angles SNP and NSP; let the curve locus of P be constructed. Prove that if a particle be placed at any point of the curve and acted on by two forces, one repulsive from N and equal to $\mu\,(\text{NP})^{-2}$, and the other attractive towards S and

equal to $\mu(\mathrm{SP})^{-2}$, the resultant force is, at every position of the particle, directed along the tangent to the curve.

I. *Solution by* Rev. J. L. KITCHIN, M.A.; J. O. JELLY, M.A.; *and others.*

Let PT be the tangent to the locus of P; then, if SNP $= \theta$, NSP $= \theta'$, and if the arc-element ds be measured in the direction in which θ increases and θ' diminishes, we have



$$\sin \mathrm{NPT} = \frac{r d\theta}{ds}, \ \sin \mathrm{SPT} = -\frac{r' d\theta'}{ds}.$$

And, since the normal component of the forces vanishes, $\frac{\mu}{r^2} \sin \mathrm{NPT} = \frac{\mu}{r'^2} \sin \mathrm{SPT}$.

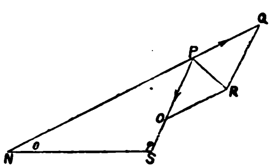
Whence $\frac{d\theta}{r} + \frac{d\theta'}{r'} = 0$. Also, by the question,

$$\cos\theta + \cos\theta' = \text{const.}, \quad r\sin\theta = r'\sin\theta' \ \ldots\ldots\ldots\ldots\ldots(1, 2).$$

Differentiating (1), $\sin\theta\, d\theta + \sin\theta'\, d\theta' = 0$; hence, by (2), $\frac{d\theta}{r} + \frac{d\theta'}{r'} = 0$, which is the condition that the resultant force should act along the tangent. [We have taken this problem from MINCHIN's *Statics.*]

II. *Solution by* R. RAWSON; GABRIEL TORELLI; *and others.*

Let P be a point in the curve locus; produce NP to Q, where PQ represents the force $\mu\,(\mathrm{NP})^n$, and PO represents the force $\mu\,(\mathrm{SP})^n$; and complete the parallelogram PQRO; then PR represents the resultant in magnitude and direction. Put NP $= \gamma$, then (r, θ) are the polar coordinates of the curve locus to which PR is to be a tangent. By TODHUNTER's *Diff. Calc.*, page 299,



$$\mathrm{NPR} = \tan^{-1}\frac{r d\theta}{d} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

also $$\mathrm{PS}\sin\phi = r\sin\theta, \quad r\sin(\theta+\phi) = \mathrm{NS}\sin\phi \ldots\ldots\ldots\ldots\ldots(2).$$

Differentiating (2), regarding NS as constant, we have

$$r\frac{d\theta}{dr} = \frac{\sin(\theta+\phi)}{\frac{\sin\theta}{\sin\phi}\cdot\frac{d\phi}{d\theta} - \cos(\theta+\phi)}.$$

(1) becomes $$\mathrm{NPR} = \tan^{-1}\left\{\frac{\sin(\theta+\phi)}{\frac{\sin\theta}{\sin\phi}\cdot\frac{d\phi}{d\theta} - \cos(\theta+\phi)}\right\} \ \ldots\ldots\ldots\ldots(3).$$

By the parallelogram of forces,

$$\frac{\sin \mathrm{ORP}}{\sin \mathrm{OPR}} = \frac{\mathrm{OP}}{\mathrm{OR}} = \frac{\mu\,(\mathrm{SP})^n}{\mu\,(\mathrm{PN})^n} = \frac{\sin^n\theta}{\sin^n\phi} \ \ldots\ldots\ldots\ldots(4),$$

and $$\mathrm{ORP} + \mathrm{OPR} = \theta + \phi.$$

From (4), $\quad \text{OPR} = \tan^{-1}\left\{\dfrac{\sin(\theta+\phi)}{\dfrac{\sin^n\theta}{\sin^n\phi} + \cos(\theta+\phi)}\right\}$ ....(5).

And, because $180° - \theta - \phi = \text{NPR} - \text{OPR}$, then

$$180° - \theta - \phi = \tan^{-1}\left\{\frac{\sin(\theta+\phi)}{\dfrac{\sin\theta}{\sin\phi}\cdot\dfrac{d\phi}{d\theta} - \cos(\theta+\phi)}\right\} - \tan^{-1}\left\{\frac{\sin(\theta+\phi)}{\dfrac{\sin^n\theta}{\sin^n\phi} + \cos(\theta+\phi)}\right\} \quad ......(6),$$

or, by obvious reductions, $\dfrac{d\phi}{\sin^{n-1}\phi} + \dfrac{d\theta}{\sin^{n-1}\theta} = 0$ ....(7).

Equation (7) is, therefore, the differential relation between the base angles θ and ϕ, which determine the curve locus.

If $n = -2$, then $\quad \cos\phi + \cos\theta = C$..................................(8).

This equation is the property enunciated in the question, and determines the curve locus.

When $n = -3$, the curve locus is $\sin 2\theta + \sin 2\phi - 2(\theta+\phi) = C$;
when $n = -4$, the curve locus is $\cos 3\theta + \cos 3\phi - 9(\cos\theta + \cos\phi) = C$.

5539. (By Professor Wolstenholme, M.A.)—If m be any positive quantity, and $a > b > c > d$, prove that

$$\int_b^a \frac{\{(a-x)(x-b)(x-c)(x-d)\}^{m-1}}{\left(\dfrac{(x-b)(x-c)}{b-c} + \dfrac{(a-x)(x-d)}{a-d}\right)^{2m-1}}\,dx$$

$$= \int_d^c \frac{\{(a-x)(b-x)(c-x)(x-d)\}^{m-1}}{\left(\dfrac{(b-x)(c-x)}{b-c} + \dfrac{(a-x)(x-d)}{a-d}\right)^{2m-1}}\,dx.$$

Solution by R. Rawson; Prof. Morel; *and others.*

Let us consider the general integral

$$u = \int_\beta^\alpha \frac{\{(x-a)(x-b)(x-c)(x-d)\}^{m-1}}{\left\{\dfrac{(x-b)(x-c)}{b-c} - \dfrac{(x-a)(x-d)}{a-d}\right\}^{2m-1}}\cdot dx \quad ....(1),$$

which represents each of the integrals in the question. Transforming (1) by the relation $\quad x = \dfrac{c(a-b) + a(b-c)\,y}{a-b+(b-c)\,y}$ ....(2),

we have, if $r(b-c)(a-d) = (a-b)(c-d)$,

$$u = \{(b-c)(a-d)\}^{m-1}\int_{\frac{(\beta-c)(a-d)r}{(a-\beta)(c-d)}}^{\frac{(\alpha-c)(a-d)r}{(a-\alpha)(c-d)}} \frac{\{(1-y)\,y\,(y+r)\}^{m-1}}{(y^2+r)^{2m-1}}\cdot dy \quad ...(3).$$

Let u_1, u_2 be the respective values of u when ($\alpha = a$, $\beta = b$, and $\alpha = c$, $\beta = d$); then, from (3),

$$u_1 = \{(b-c)(a-d)\}^{m-1}\int_1^{\infty}\frac{\{(1-y)\,y\,(y+r)\}^{m-1}}{(y^2+r)^{2m-1}}\,dy \quad\text{.........(4)},$$

$$u_2 = \{(b-c)(a-d)\}^{m-1}\int_{-r}^{0}\frac{\{1-y)\,y\,(y+r)\}^{m-1}}{(y^2+r)^{2m-1}}\,dy \quad\text{.........(5)}.$$

In (4), put $y = -\dfrac{r}{z}$, therefore $dy = \dfrac{r}{z^2}\,dz$; when $y = \infty$, therefore z = zero; when $y = 1$, therefore $z = -r$. Substituting these values in (4), we have

$$u_1 = \{(b-c)(a-d)\}^{m-1}\int_{-r}^{\infty}\frac{\left\{-\frac{r}{z}\left(1+\frac{r}{z}\right)\left(r-\frac{r}{z}\right)\right\}^{m-1}.\ \frac{r}{z^2}}{\left(\frac{r^2}{z^2}+r\right)^{\cdot}}\,dz$$

$$= \{(b-c)(a-d)\}^{m-1}\int_{-r}^{\infty}\frac{\{(1-z)(z+r)\}^{m-1}.\ z^{m-1}}{(z^2+r)^{2m-1}}\,dz$$

$$= \{(b-c)(a-d)\}^{m-1}\int_{-r}^{\infty}\frac{\{(1-z)\,z\,(z+r)\}^{m-1}}{(z^2+r)^{2m-1}}\,dz = u_2.$$

The above equations have been obtained without imposing any restrictions upon (m). On this account it seems to me that the adjective "*positive*" should be withdrawn by the learned proposer.

When $m = \frac{1}{2}$ the theorem gives

$$\int_b^a\frac{dx}{\{(x-a)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}} = \int_d^c\frac{dx}{\{(x-a)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}}\quad\text{...(6)}.$$

This theorem evidently contains Question 5508, proposed by me in December, 1877. I cannot say whether (6) has been noticed by any of the numerous writers on Elliptic Functions.

It may be stated that the quadric integral can be always reduced to a cubic integral.

[Professor Wolstenholme remarks that the reason he inserted the condition m positive was that any integral $\int_a^b (x-h)^{m-1}\,F(x)\,dx$, where $F(x)$ does not become infinite between a and b, is infinite when h is between or at a and b, and m is negative. He states that the work by which he obtained this result was first suggested by Question 5508.]

5449. (By J. L. McKenzie, B.A.)—A line drawn from the common centre of two concentric and coaxial ellipses cuts one conic in A and the other in B; prove that the locus of the harmonic conjugate of O with respect to A and B is the quartic

$$[x^2(a^{-2}-a'^{-2})+y^2(b^{-2}-b'^{-2})]^2-8\,[x^2(a^{-2}+a'^{-2})+y^2(b^{-2}+b'^{-2})]+16=0.$$

Solution by S. JOHNSTON ; H. MURPHY ; *and others.*

Let ρ_1, ρ_2, R be the central radii vectores to the points A, B, and the harmonic conjugate of O ; then we have

$$\frac{2}{R}=\frac{1}{\rho_1}+\frac{1}{\rho_2}; \text{ therefore } \left[\frac{4}{R^2}-\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right]^2=\frac{4}{\rho_1^2\rho_2^2},$$

or
$$\left(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right)^2-\frac{8}{R^2}\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right)+\frac{16}{R^4}=0.$$

But
$$\frac{1}{\rho_1^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},\qquad \frac{1}{\rho_2^2}=\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}.$$

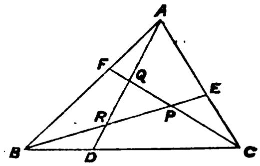
Substituting these values, multiplying across by R^4, and substituting $x = R\cos\theta$, $y = R\sin\theta$, we evidently obtain the equation in the question.

5724. (By C. LEUDESDORF, M.A.)—D, E, F are points lying on the sides BC, CA, AB respectively of a triangle; and BE, CF meet in P; CF, AD in Q; and AD, BE in R; prove that

$$\left(\frac{AE\,.\,BF\,.\,CD}{AF\,.\,BD\,.\,CE}\right)^{\frac{1}{2}}-\left(\frac{AF\,.\,BD\,.\,CE}{AE\,.\,BF\,.\,CD}\right)^{\frac{1}{2}}=\left\{\frac{(ABC)^2\,(PQR)}{(PBC)\,(QCA)\,(RAB)}\right\}^{\frac{1}{2}},$$

where (ABC), (PQR), &c. denote the areas of the triangles ABC, PQR, &c.

Solution by E. W. SYMONS; A. BUCHHEIM; D. EDWARDES; *and others.*



Let the areal equations of AD, BE, CF be $\beta = \kappa_1\gamma$, $\gamma = \kappa_2 a$, $a = \kappa_3\beta$; then the coordinates of P, Q, R are

$$\left(\lambda_1,\ \frac{\lambda_1}{\kappa_3},\ \lambda_1\kappa_2\right);\ \left(\lambda_2\kappa_3,\ \lambda_2,\ \frac{\lambda_2}{\kappa_1}\right);\ \left(\frac{\lambda_3}{\kappa_2},\ \lambda_3\kappa_1,\ \lambda_3\right).$$

Therefore, by WHITWORTH'S *Modern Geometry*, we have

$$(PQR)=(ABC)\begin{vmatrix}\lambda_1, & \lambda_2\kappa_3, & \lambda_3\kappa_2^{-1}\\ \lambda_1\kappa_3^{-1}, & \lambda_2, & \lambda_3\kappa_1\\ \lambda_1\kappa_2, & \lambda_2\kappa_1^{-1} & \lambda_3\end{vmatrix};\qquad \begin{aligned}(PBC)&=(ABC)\,\lambda_1,\\ (QCA)&=(ABC)\,\lambda_2,\\ (RAB)&=(ABC)\,\lambda_3;\end{aligned}$$

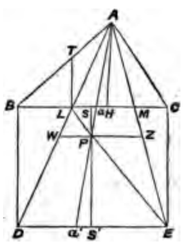
therefore
$$\left\{\frac{(ABC)^2\,.\,(PQR)}{(-)\,(-)\,(-)}\right\}=\left\{\begin{vmatrix}1, & \kappa_3, & \kappa_2^{-1}\\ \kappa_3^{-1}, & 1, & \kappa_1\\ \kappa_2, & \kappa_1^{-1}, & 1\end{vmatrix}\right\}^{\frac{1}{2}}$$

$$=(\kappa_1\kappa_2\kappa_3)^{\frac{1}{2}}\sim(\kappa_1\kappa_2\kappa_3)^{-\frac{1}{2}}=\left(\frac{CD}{BD}\cdot\frac{AE}{CE}\cdot\frac{BF}{AF}\right)^{\frac{1}{2}}\sim\left(\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}\right)^{\frac{1}{2}},$$

therefore, &c.

5512. (By R. Tucker, M.A.)—BDEC is a square drawn on BC, outside the triangle ABC. AD, AE cut BC in L, M, and EL, DM intersect in P. Prove that, if Q, R are the corresponding points for the sides AC, AB, then AP, BQ, CR cointersect. PS is the perpendicular on BC; ES produced meets AB in T; LT is the side of the inscribed square (on BC) of ABC; and PS is the side of the inscribed square (on BC) of BCT. If a parallel to BC through P cut AD, AE in W, Z, prove that WP = PZ = PS.

I. *Solution by* D. Edwardes; A. Escott, M.A; *and others.*



The points L, M are the angular points of the inscribed square (on BC) of ABC. Hence, drawing LT perpendicular to BC, LT is a side of the inscribed square. We have then to shew that T, S, E are in one straight line, and that PS is a side of the inscribed square (on BC) of BTC. Let AP cut BC, DE in a, a'.

Then, by the harmonic properties of the quadrilateral LDEM, aa' is divided harmonically in A, P;

therefore $\quad P a \,.\, A a' = A a \,.\, P a'.$

Drawing AH perpendicular to BC, we have

$$\frac{Pa}{PS} = \frac{Pa'}{PS'} = \frac{Aa}{AH} = \frac{Aa'}{AH+BC};$$

whence $\quad PS\,(2AH + BC) = BC \,.\, AH.$

Now, LT being a side of the inscribed square (on BC) of ABC,

we have $\quad LT = \dfrac{AH \,.\, BC}{AH+BC}$, whence $PS = \dfrac{BC \,.\, LT}{BC+LT}$ ....(1, 2);

which shews that PS is a side of the inscribed square (on BC) of BCT. Also $\dfrac{EL}{PE} = \dfrac{BC}{BC-PS} = \dfrac{BC+LT}{BC} = \dfrac{LT}{PS}$ from (2), or $\dfrac{LT}{LE} = \dfrac{PS}{PE}$; and PS is parallel to LT; therefore T, S, E are in one straight line. Because the side LM of the quadrilateral LDEM is parallel to side DE, the point conjugate to a' with respect to D, E is at infinity, therefore a' is middle point of DE. Hence, if WPZ be parallel to DE, WP = PZ.

From the triangle DLE, $\dfrac{WP}{BC\,(=DE)} = \dfrac{PL}{LE}$; but $\dfrac{LE}{PE} = \dfrac{LT}{PS}$, whence $\dfrac{PL}{LE} = 1 - \dfrac{PS}{LT} = 1 - \dfrac{BC}{BC+LT}$, from (2), $= \dfrac{LT}{BC+LT} = \dfrac{PS}{BC}$, from (2), therefore $\dfrac{WP}{BC} = \dfrac{PS}{BC}$, therefore WP = PZ = PS.

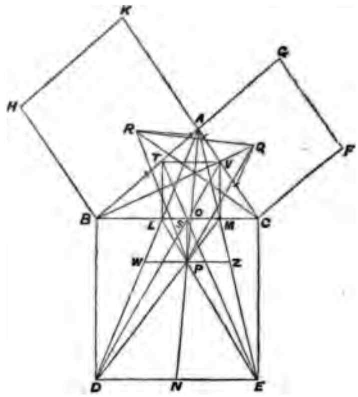
Since a is the middle point of LM, the side of the inscribed square on BC, we have $\quad \dfrac{Ba}{Ca} = \dfrac{\cot B + \frac{1}{2}}{\cot C + \frac{1}{2}}.$

Hence, if Bb, Cc be the two other similarly drawn lines, and ABC be the triangle of reference, the equations of Aa, Bb, Cc are

$$\beta\,(\cos B + \tfrac{1}{2}\sin B) - \gamma\,(\cos C + \tfrac{1}{2}\sin C) = 0,$$
$$\gamma\,(\cos C + \tfrac{1}{2}\sin C) - \alpha\,(\cos A + \tfrac{1}{2}\sin A) = 0,$$
$$\alpha\,(\cos A + \tfrac{1}{2}\sin A) - \beta\,(\cos B + \tfrac{1}{2}\sin B) = 0;$$

hence these lines meet in a point.

II. *Solution by the* PROPOSER.



$$\frac{\text{PS}}{(\text{DE or}) \text{ BD}} = \frac{\text{LM}}{\text{LM}+\text{DE}}$$

$$\therefore \quad \frac{\text{PS}}{\text{LM (or LT}_1)} = \frac{\text{DE}}{\text{LM}+\text{DE}} = \frac{\text{EP}}{\text{EL}},$$

where LT_1 is the side of the square perpendicular to BC; therefore EST_1 is a straight line, that is, T_1 and T coincide.

Again, we have

$$\frac{\text{PS}}{\text{LT}} = \frac{\text{EO}}{\text{ET}} = \frac{\text{BC}}{\text{BC}+\text{LM}\,(=\text{LT})},$$

therefore PS is a side of the square inscribed in the triangle BCT.

Above,

$$\frac{\text{PS}}{\text{LM}} = \frac{\text{EP}}{\text{EL}} = \frac{\text{WP}}{\text{LM}} = \frac{\text{PZ}}{\text{LM}}, \text{ therefore } \text{PS} = \text{PW} = \text{PZ}.$$

[The remainder of Mr. TUCKER's proof was analytical, and almost identically the same as that given in the foregoing Solution.]

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149. NOTE ON THE SOLUTION OF QUESTION 5701. *By the* EDITOR. (See p. 40 of this Volume.)

After this solution had been printed off, it appeared that the value $\sec\alpha$ for u, that is, for the series $1+\frac{1}{2}\sin^2\alpha+\frac{3}{8}\sin^4\alpha+\ldots$, followed immediately from the expansion of $(1-\sin^2\alpha)^{-\frac{1}{2}}$.

The evaluation of this series by "BOOLE's *Symbolic Method*" and an elaborate use of differential equations was, in fact, a remarkable instance of employing a steam hammer to crack a nut.

5511. (By HUGH MCCOLL, B.A.)—Given that $x+y+z=u$, $x+y=uv$, $y=uvw$, transform $\int_0^a dx \int_0^a dy \int_0^a dz\, \phi(x, y, z)$ into an integral of the form $\int du \int dv \int dw\, \psi(u, v, w)$, and determine the limits of u, v, w.

Solution by the PROPOSER.

From the given integral we get the compound statement

$$p'(x-a)\,px\,p'(y-a)\,py \times p'(z-a)\,pz.$$

Let this statement be denoted by A (see my articles on *Symbolical Language*).

Table of Limits.		
$w_1 = 1 - \frac{a}{uv}$	$v_1 = 1 - \frac{a}{u}$	$u_1 = 2a$
$w_2 = 1$	$v_2 = 1$	$u_2 = a$
$w_3 = \frac{a}{uv}$	$v_3 = \frac{a}{u}$	$u_3 = 3a$
	$v_4 = \frac{2a}{u}$	

Now $p'(x-a) = p'(uv-uvw-a) = p\left\{uv\left(w-1+\frac{a}{uv}\right)\right\}$

$$= p\left\{uv\,(w-w_1)\right\} = u_0 v_0 w_1 + u_{0'} v_{0'} w_1 + u_0 v_{0'} w_{1'} + u_{0'} v_0 w_{1'}.$$

Similarly, we get

$$px = u_{0'} v_{0'} w_{2'} + u_{0'} v_0 w_2 + u_0 v_{0'} w_2 + u_0 v_0 w_{2'},$$

Hence $p'(x-a)\,px = u_{0'} v_{0'} w_{2'.1} + u_{0'} v_0 w_{1'.2} + u_0 v_{0'} w_{1'.2} + u_0 v_0 w_{2'.1}$, omitting terms containing inconsistent compound factors, such as $u_{0'.0}$, $v_{0'.0}$.

Similarly, we get

$$p'(y-a)\,py = u_0 v_0 w_{3'.0} + u_{0'} v_{0'} w_{3'.0} + u_0 v_{0'} w_{0'.3} + u_{0'} v_0 w_{0'.3},$$

and $$p'(z-a)\,pz = u_{0'} v_{1'.2} + u_0 v_{2'.1}.$$

Multiplying together the three expansions for $p'(x-a)\,px$, $p'(y-a)\,py$, $p'(z-a)\,pz$, and omitting those terms in the product which contain inconsistent compound factors, such as $u_{0'.0}$, $v_{0'.0}$, $v_{0'.2}$, $w_{0'.2}$, &c., only one term will be found to remain, namely, $u_0 v_{2'.1.0} w_{3'.2'.1.0}$.

Hence $$A = u_0 v_{2'.1.0} w_{3'.2'.1.0}.$$

Applying Rules 3 and 4 to the w-statements, we get

$$v_0 w_{3'.2'.1.0} = v_0 (w_{3'} v_3 + w_{2'} v_{3'})(w_1 v_3 + w_0 v_{3'})$$
$$= v_0 (w_{3'.1} v_3 + w_{2'.0} v_{3'}),$$

omitting the terms which contain the inconsistent compound factor $v_{3'.3}$.

Hence $$A = u_0 (w_{3'.1} v_{2'.3.1.0} + w_{2'.0} v_{3'.2'.1.0}).$$

Applying Rule 5 to the w-statements, we get

$$v_0 w_{3'.1} = v_0 w_{3'.1}\, p(w_3 - w_1) = v_{4'.0} w_{3'.1}$$

and $$w_{2'.0} = w_{2'.0}\, p(w_2) = w_{2'.0} \times 1 = w_{2'.0}.$$

Substituting, we get

$$A = u_0 (w_{3'.1} v_{4'.2'.3.1.0} + w_{2'.0} v_{3'.2'.1.0}).$$

Applying Rules 3 and 4 to the v-statements, we get

$$u_0 v_{4'.2'} = u_0 (v_{4'} u_1 + v_{2'} u_{1'}), \qquad u_0 v_{3.1.0} = u_0 (v_3 u_{1'} + v_1 u_1),$$
$$u_0 v_{3'.2'} = u_0 (v_{3'} u_2 + v_{2'} u_{2'}), \qquad u_0 v_{1.0} = u_0 (v_1 u_2 + v_0 u_{2'}).$$

Hence $$A = u_0 \{ w_{3'.1} (v_{4'} u_1 + v_2 u_{1'}) (v_3 u_{1'} + v_1 u_1) \\ + w_{2'.0} (v_{3'} u_2 + v_{2'} u_{2'}) (v_1 u_2 + v_0 u_{2'}) \}.$$

Multiplying, omitting zero terms, and observing that $u_{1.0} = u_1$, and that $u_{2.0} = u_2$, we get

$$A = w_{3'.1} (v_{4'.1} u_1 + v_{2'.3} u_{1'.0}) + w_{2'.0} (v_{3'.1} u_2 + v_{2'.0} u_{2'.0}).$$

Applying Rule 5 to the v-statements, we get

$$u_1 v_{4'.1} = u_{3'.1} v_{4'.1}, \qquad u_0 v_{2'.3} = u_{2.0} v_{2'.3} = u_2 v_{2'.3},$$
$$u_2 v_{3'.1} = u_{1'.2} v_{3'.1}, \qquad v_{2'.0} = v_{2'.0} \times 1 = v_{2'.0},$$

Substituting, we get

$$A = w_{3'.1} (v_{4'.1} u_{3'.1} + v_{2'.3} u_{1'.2}) + w_{2'.0} (v_{3'.1} u_{1'.2} + v_{2'.0} u_{2'.0}).$$

Here the process terminates, for the application of Rule 5 to the u-statements will introduce no fresh factors into any of the terms.

The *limits* of the new variables being thus determined, the rest of the solution follows from the usual formula (see TODHUNTER'S *Integral Calculus*, p. 234, Art. 247). From this formula we get $dx\,dy\,dz = u^2 v\,du\,dv\,dw$. Let $\phi_1 (u, v, w)$ denote what $\phi (x, y, z)$ becomes when for x, y, z we substitute their values in terms of u, v, w. The last expression for A will then denote the required transformed integral, provided we consider $w_{3'.1} v_{4'.1} u_{3'.1}$ as an abbreviation for the definite integral

$$\int_{u_1}^{u_3} du \int_{v_1}^{v_4} dv \int_{w_1}^{w_3} dw \,.\, u^2 v \,.\, \phi_1 (u, v, w),$$

and put a similar interpretation on the other three terms.

[The method to which Mr. McCOLL has given the title of *Symbolical Language* was presented by Professor CROFTON to the Mathematical Society, at its meeting last November, under the title of *The Calculus of Equivalent Statements*; but the latter paper contains full explanations and illustrations for which we could not find space in the *Educational Times*.]

5334. (By CHRISTINE LADD.)—In a spherical triangle, given a, b, B; express the sine and the cosine of c and C in terms of the data.

Solution by the PROPOSER.

Giving the trigonometrical interpretation of Art. 51 to the formula of Art. 86 CAYLEY'S *Elliptic Functions*, we have

$$\sin c = \sin b \cos a \cos A \pm \sin a \cos b \cos B \,(\div),$$
$$\cos c = \cos a \cos b \mp \sin a \sin b \cos A \cos B \,(\div),$$
$$\cos C = \sin A \sin B \cos a \cos b \sin a \sin b \mp \cos A \cos B \,(\div),$$

the denominator being in each case

$$1 - \sin^2 A \sin^2 b = 1 - \sin^2 a \sin^2 B;$$

and, since $\sin C = k \sin c$, we have, with the same denominator,

$$\sin C = \sin B \cos a \cos A \pm \sin A \cos b \cos B \; (\div).$$

[The useful notation $(\div)$ is used by Professor CAYLEY (see Art. 13 of his *Elliptic Functions*) to show that the function which it is affixed to is a fraction whose denominator is stated afterwards. In our own limited space this notation will be so convenient, that we shall be glad if all our correspondents will hereafter make use of it.

Miss LADD's expressions for $\sin c$, $\cos c$, $\cos C$ are noticed in LEGENDRE (Vol. I., p. 22), and are at once obtained from the formula

$$c = \tan^{-1}(\tan a \cos B) + \tan^{-1}(\tan b \cos A),$$

which expresses that c is the sum of the two parts into which it is divided by the foot of the perpendicular let fall upon it from the opposite angle C.]

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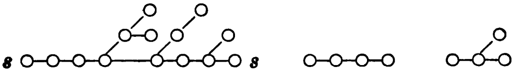
573. (By Professor SYLVESTER, F.R.S.)—Suppose the ramification of a tree to be subject to the law that each joint may throw out a single branch or two branches, but never more than two. Then, if n, the number of joints, be 1, 2, 3, 4, 5, 6, 7, 8, &c., N, the corresponding number of distinct forms of arborescence, will be found to be 1, 1, 1, 2, 2, 4, 6, 11, &c. Find the general relation between n and N.

[Professor SYLVESTER remarks that this problem is of considerable importance in chemistry. If the number of branches be allowed to be one, two, or three (instead of only one or two, as in the Question), the problem is that of finding how many *isomeric compounds* of the hydrocarbons $C_n H_{2n+2}$ can be found for a given value of n, according to the law of Kekulé and Crum-Brown, expressed by the word "Tetratomicity." This problem, which is simpler, belongs to the Hydro-Boron class $B_n H_{n+2}$ (which, however, only exist hypothetically), and would come under the law of *Triatomicity*. The Tetratomic Numbers are 1, 1, 1, 2, 3, 6, 9, 18, 34, &c., where, for example, 34 is the possible number of Hydrocarbons represented by the formula C_9H_{20}.]

Solution by SEPTIMUS TEBAY, B.A.



(1) (2) (3)

If we draw any tree, as (1), we see that it consists of a main stem, ss, and branches of the forms (2), (3).

Let m be the number of joints on the main stem, and a_1 the number of joints on a simple branch, as (2). This branch can have $m - 2a_1$ positions on the main stem, and no more, for then the branch a_1 would form part of a new stem greater than m, which would lead to a repetition of forms. Let $a_2, a_3, \ldots a_r$ be other branches from the main stem, and suppose $m > 2a_1 > 2a_2 \ldots > 2a_r$; then these branches can have

$$m - 2a_1, \quad m - 2a_2 - 1, \quad m - 2a_3 - 2, \quad \ldots \quad m - 2a_r - r + 1$$

positions on the main stem; and therefore the whole number of forms

$$= (m-2a_1)(m-2a_2-1)\dots(m-2a\ -r+1) \text{ ....(A).}$$

If a_1, a_2, &c. be all different, then, in the selection of distinct forms of arborescence, it will be seen, on glancing from each end of the main stem, that all the forms in (A) are repeated. If $a_1, a_2, \dots a_i$ are repeated $\alpha_1, \alpha_2, \dots \alpha_i$ times, (A) becomes, putting $\alpha_1+\alpha_2+\dots+\alpha_i=\sigma_i$.

$$\frac{(m-2a_1)\,!}{\alpha_1\,!\,(m-2a_1-\sigma_1)\,!}\cdot\frac{(m-2a_2-\sigma_1)\,!}{\alpha_2\,!\,(m-2a_2-\sigma_2)\,!}\cdots\frac{(m-2a_i-\sigma_{i-1})\,!}{\alpha_i\,!\,(m-2a_i-\sigma_i)\,!}\text{ ......(P).}$$

If m be odd, and one of the members α_1, α_2, &c. also odd, and the others even, there will be non-repeating forms in (P). Let one of the odd branches occupy the middle joint of m; then, omitting this branch, there will be $\frac{1}{2}\alpha_1$, $\frac{1}{2}\alpha_2$, &c. branches on each side of the middle branch, presenting

$$\frac{(m'-a_1)\,!}{\frac{1}{2}\alpha_1\,!\,(m'-a_1-\frac{1}{2}\sigma_1)\,!}\cdot\frac{(m'-a_2-\frac{1}{2}\sigma_1)\,!}{\frac{1}{2}\alpha_2\,!\,(m'-a_2-\frac{1}{2}\sigma_2)\,!}\cdots\frac{(m'-a_2-\frac{1}{2}\sigma_{i-1})\,!}{\frac{1}{2}\alpha_i\,!\,(m'-a_i-\frac{1}{2}\sigma_i)\,!}\text{......(Q)}$$

non-repeating forms; where $m' = \frac{1}{2}\left[m-\frac{1}{2}\{1-(-1)^m\}\right]$.

If m be odd, and α_1, α_2, &c. all even, the non-repeating form is (Q).

If m be even, and α_1, α_2, &c. all even, the non-repeating form is (Q).

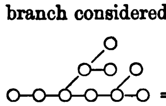
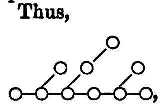
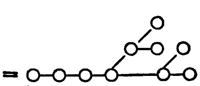
If m be even, and one or more of α_1, α_2, &c. odd, every form in (P) is repeated. Hence, recapitulating, we have for simple branches

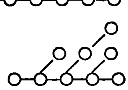
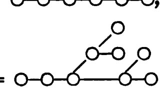
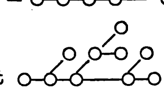
$$N_m = \tfrac{1}{2}\Sigma\,[P+Q],\ m \text{ odd, one only of } \alpha_1, \alpha_2, \text{ \&c. odd;}$$
$$N_m = \tfrac{1}{2}\Sigma\,[P+Q],\ m \text{ odd, } \alpha_1, \alpha_2, \text{ \&c. all even;}$$
$$N_m = \tfrac{1}{2}\Sigma\,[P+Q],\ m \text{ even, } \alpha_1, \alpha_2, \text{ \&c. all even;}$$
$$N_m = \tfrac{1}{2}\Sigma\,[P],\ m \text{ even, some of } \alpha_1, \alpha_2, \text{ \&c. odd;}$$

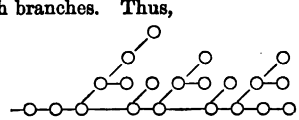
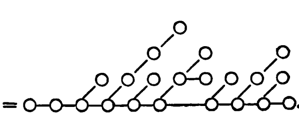
where Σ embraces every form of the general type

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_i a_i = n-m.$$

It must be observed that a_1, a_2, &c. can each be treated as sub-values of n, when number and position are suitable; with this difference, that the end of the sub-stem being closed, there will be no repeated forms. Also, with regard to compound branches, the general rule is that the number of joints from the extremity of any branch to either end of the main stem must not exceed m. Even when this number is equal to m, if the branch be a compound one, and the end of the stem open, there will be a repetition of forms. This implies that all the other branches are on one side of the branch considered. Thus,

 = , but not = 

Also,  = ; whilst  is not changed. Further, it is plain that both ends of the stem may be open to such branches. Thus,

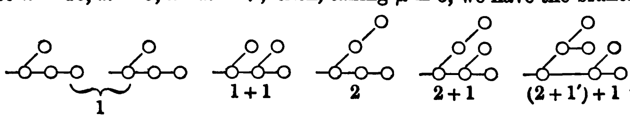
 = .

The correct number of forms can be obtained by considering these as simple branches, deducting the number of repeated forms, which can always be found by the above formulæ.

The number of distinct branches on any sub-stem (μ) may be computed from the formula

$$\frac{(\mu-a_1)\,!}{a_1\,!\,(\mu-a_1-\sigma_1)\,!}\cdot\frac{(\mu-a_2-\sigma_1)\,!}{a_2\,!\,(\mu-a_2-\sigma_2)\,!}\cdots\frac{(\mu-a_i-\sigma_{i-1})\,!}{a_i\,!\,(\mu-a_i-\sigma_i)\,!}\ \ldots\ldots\ldots(\mathrm{R}).$$

In this way branches can be formed, and added to the main stem. Thus: Let $n = 16$, $m = 9$, $n-m = 7$; then, taking $\mu = 3$, we have the branches



the accent denoting a second sub-branch.

The preceding formulæ can be applied to the s-tomic theory by combining the simple branches in groups not exceeding $s-1$, and taking the largest as the representative branch of the group. Thus, let $s = 3$, $r = 4$; then, a_1, a_2, a_3, a_4 being all different,

$a_1+a_2+a_3+a_4$ gives $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_3-2)(m-2a_4-3)$ forms

$a_1a_2+a_3+a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_3-1)(m-2a_4-2)$ forms.

$a_1a_3+a_2+a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_4-2)$ „

$a_1a_4+a_2+a_3$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_3-2)$ „

$a_1+a_2+a_3a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_3-2)$ „

$a_1+a_2a_3+a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_4-2)$ „

$a_1+a_2a_4+a_3$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)(m-2a_3-2)$ „

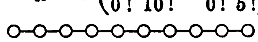
$a_1a_2+a_3a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_3-1)$ forms.

$a_1a_3+a_2a_4$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)$ „

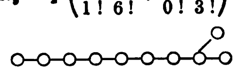
$a_1a_4+a_2a_3$ „ $\frac{1}{2}(m-2a_1)(m-2a_2-1)$ „

Examples.

Let $n = 10$, $m = 10$; $N_{10} = \frac{1}{2}\left(\frac{10\,!}{0\,!\;10\,!}+\frac{5\,!}{0\,!\;5\,!}\right) = 1.$

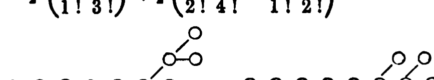


$m = 9$, $a_1 = 1$; $N_9 = \frac{1}{2}\left(\frac{7\,!}{1\,!\;6\,!}+\frac{3\,!}{0\,!\;3\,!}\right) = 4.$



$m = 8$, $a_1+a_2 = 2 = 1+1$;

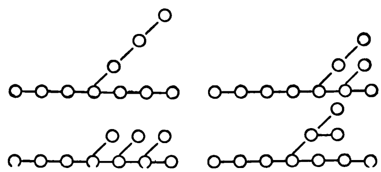
$$N_8 = \frac{1}{2}\left(\frac{4\,!}{1\,!\;3\,!}\right)+\frac{1}{2}\left(\frac{6\,!}{2\,!\;4\,!}+\frac{3\,!}{1\,!\;2\,!}\right) = 2+9 = 11.$$



$m = 7$, $a_1+a_2+a_3 = 3 = 2+1 = 1+1+1 = (2+1')$,
where the accent denotes a compound branch;

$$N_7 = \tfrac{1}{2}\left(\frac{1\,!}{1\,!\;0\,!} + \frac{0\,!}{0\,!\;0\,!}\right) + \tfrac{1}{2}\left(\frac{3\,!}{1\,!\;2\,!}\cdot\frac{4\,!}{1\,!\;3\,!}\right) + \tfrac{1}{2}\left(\frac{5\,!}{3\,!\;2\,!} + \frac{2\,!}{1\,!\;1\,!}\right)$$

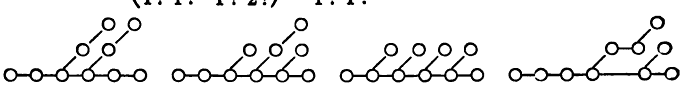
$$+\tfrac{1}{2}\left(\frac{3\,!}{1\,!\;2\,!} + \frac{1\,!}{0\,!\;1\,!}\right) - \frac{4\,!}{0\,!\;4\,!} = 1+6+6+2-1 = 14.$$



$m = 6$, $a_1 + \&c. = 4 = 2+2 = 2+1+1 = 1+1+1+1 = (2+1)+1$;

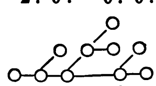
$$N_6 = \tfrac{1}{2}\left(\frac{2\,!}{2\,!\;0\,!} + \frac{1\,!}{1\,!\;0\,!}\right) + \tfrac{1}{2}\left(\frac{2\,!}{1\,!\;1\,!}\cdot\frac{3\,!}{2\,!\;1\,!}\right) + \tfrac{1}{2}\left(\frac{4\,!}{4\,!\;0\,!} + \frac{2\,!}{2\,!\;0\,!}\right)$$

$$+\tfrac{1}{2}\left(\frac{2\,!}{1\,!\;1\,!}\cdot\frac{3\,!}{1\,!\;2\,!}\right) - \frac{2\,!}{1\,!\;1\,!} = 1+3+1+3-2 = 6.$$

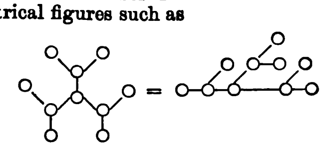


$m = 5$, $a_1 + \&c. = 5 = (2+1')+1+1$.

$$N_5 = \tfrac{1}{2}\left(\frac{1\,!}{1\,!\;0\,!}\cdot\frac{2\,!}{2\,!\;0\,!} + \frac{0\,!}{0\,!\;0\,!}\cdot\frac{1\,!}{1\,!\;0\,!}\right) = 1.$$

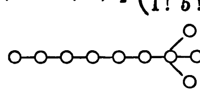


Therefore N = 37. The number 10 is of the form $n = 3\,(2^j - 1) + 1$, furnishing symmetrical figures such as



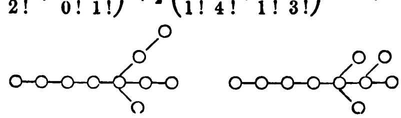
To find the tetratomic number we have, when

$$m = 8,\ 2 = 1, 1;\ \tfrac{1}{2}\left(\frac{6\,!}{1\,!\;5\,!}\right) = 3;$$



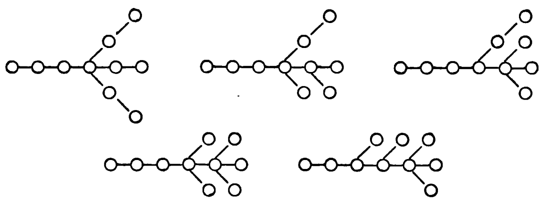
$$m = 7,\ 3 = 2{,}1 = 1, 1+1;$$

$$\tfrac{1}{2}\left(\frac{3\,!}{1\,!\;2\,!} + \frac{1\,!}{0\,!\;1\,!}\right) + \tfrac{1}{2}\left(\frac{5\,!}{1\,!\;4\,!}\cdot\frac{4\,!}{1\,!\;3\,!}\right) = 2 + 10 = 12.$$



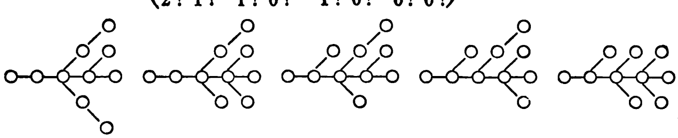
$$m = 6,\ 4 = 2, 2 = 2, 1+1 = 2+1, 1 = 1, 1+1, 1 = 1, 1+1+1\ ;$$

$$\tfrac{1}{2}\left(\frac{2\,!}{1\,!\ 1\,!}\right)+\tfrac{1}{2}\left(\frac{2\,!}{1\,!\ 1\,!}\cdot\frac{3\,!}{1\,!\ 2\,!}\right)+\tfrac{1}{2}\left(\frac{2\,!}{1\,!\ 1\,!}\cdot\frac{3\,!}{1\,!\ 2\,!}\right)+\tfrac{1}{2}\left(\frac{4\,!}{2\,!\ 2\,!}+\frac{2\,!}{1\,!\ 1\,!}\right)$$
$$+\tfrac{1}{2}\left(\frac{4\,!}{1\,!\ 3\,!}\cdot\frac{3\,!}{1\,!\ 2\,!}\right) = 1+3+3+4+6 = 17.$$



$$m=5,\ 5 = 2, 2+1 = 2, 1+1, 1 = 2, 1+1+1 = 2+1, 1+1 = 1, 1+1, 1+1.$$

$$\tfrac{1}{2}\left(\frac{1\,!}{1\,!\ 0\,!}\cdot\frac{2\,!}{1\,!\ 1\,!}\right)+\tfrac{1}{2}\left(\frac{1\,!}{1\,!\ 0\,!}\cdot\frac{2\,!}{1\,!\ 1\,!}\right)$$
$$+\tfrac{1}{2}\left(\frac{1\,!}{1\,!\ 0\,!}\cdot\frac{2\,!}{2\,!\ 0\,!}+\frac{0\,!}{0\,!\ 0\,!}\cdot\frac{1\,!}{0\,!\ 0\,!}\right)+\tfrac{1}{2}\left(\frac{1\,!}{1\,!\ 0\,!}\cdot\frac{2\,!}{1\,!\ 1\,!}\cdot\frac{1\,!}{1\,!\ 0\,!}\right)$$
$$+\tfrac{1}{2}\left(\frac{3\,!}{2\,!\ 1\,!}\cdot\frac{1\,!}{1\,!\ 0\,!}+\frac{1\,!}{1\,!\ 0\,!}\cdot\frac{0\,!}{0\,!\ 0\,!}\right) = 1+1+1+1+2 = 6.$$



We have thus 38 additional forms, making the tetratomic number 75.

It is convenient to work out particular examples in this way, inasmuch as the relations between m and a_1, a_2, &c. can at once be taken advantage of.

[Mr. TEBAY states that, in forwarding his work on the "tree problem," he gives it in an illustrated form, which seems to be necessary, as the subject is a particular one and needs attention. It may be likened to a line of rails between two termini, having intermediate stations, with branch lines and sub-branches *ad libitum*, within certain limits. Mr. TEBAY adds that he has not found Prof. SYLVESTER's tetratomic number 34, but makes it 35, both by *analysis* and by *synthesis*.]

5743. (By T. COTTERILL, M.A.)—In a system of three line-pairs in a plane, the lines cutting them in involution envelop a class cubic, and the locus of the double points of the involution is an order cubic. When three of the lines (one of each pair) pass through one point, and the other three through another point, show that one curve breaks up into a conic and line, and the other into three points.

I. *Solution by* Dr. Hirst, F.R.S.

Each of the six lines of the three line-pairs a, a'; b, b'; c, c' obviously, cuts the system in involution, and exclusive of the lines a, a' there is but one line passing through the point (aa') which can do so. For (aa') will clearly be a double point of the involution on any such line, and the other double point will necessarily lie on the polar of (aa') with respect to the line-pair b, b', as well as on its polar with respect to the line-pair c, c'; in other words, it will coincide with the intersection of these polars. This proves that the envelope of lines which cut the system in involution is of the third class.

Again, exclusive of the two double points of the involution on a itself, there is but one double point situated on a; viz., the point (aa') already mentioned. The locus of double points, therefore, is of the third order.

When a, b, c pass through a point P, and a', b', c' through a point Q, the line $\overline{PQ}$ will obviously cut the system in an involution, consisting of two points only, whose double points are harmonic conjugates relative to P and Q, but otherwise indeterminate. The line $\overline{PQ}$ therefore is, in this case, a constituent part of the cubic locus of double points; the remaining constituent is a conic (C) which passes through the points (aa'), (bb'), and (cc'), as we know, as well as through P and Q. The truth of the last statement will be evident on observing that *every* line through P (or Q) cuts the system in an involution *both* of whose double points are coincident with P (or Q). From this it follows, further, that the points P and Q are constituent parts of the third-class envelope of lines cutting the given line-pairs in involution, the remaining constituent being a third point R, which may be readily proved to be the pole of $\overline{PQ}$ relative to the conic (C). In fact, P, Q, (aa'), and (bb') being four points on the conic (C), the polar of the intersection of $\overline{PQ}$ and $\overline{(aa')\,(bb')}$ must pass through the intersections (ab') and $(a'b)$, and on that account be a line which cuts the given system in involution. But as polar of a point in $\overline{PQ}$ this line also passes through the pole of the latter relation to (C), and the same being true of the line through (bc') and $(b'c)$, as well as of that through (ca') and $(c'a)$, both of which also cut the system in involution, it follows that R coincides with the pole of $\overline{PQ}$.

The Theorems of the question occur, with many other very beautiful ones, in the Theory of Conics forming a *reseau* (*Netz*), as treated by Cremona, Schröter, Reye, and others.

II. *Solution by* Professor Townsend, F.R.S.

These two properties are true, not only for a triad of angles, but more generally for any triad of conics, having a common chord of intersection; the two cubics in question (see Salmon's *Conic Sections*, Ed. 5, Art. 388) breaking up for them, the former into the two common points and the radical centre of the triad, and the latter into the common chord and the conic containing it and the six points of contact with the triad of the six tangents to them from their radical centre. In the particular case of three angles, as proposed in the question, the conic of the latter case is that containing the common chord and their three vertices, and the tangents to it at the common points intersect, as in the general case, at the radical centre of the triad.

III. *Solution by the* Rev. F. D. THOMSON, M.A.

1. Let A, A′, B, B′, C, C′ be the six lines, and let a line L cut them in the points a, a', b, b', c, c' in involution; then we have

$$\{abca'\} = \{a'b'c'a\} \text{ or } ab \, . \, ca' \, . \, b'c' + a'b' \, . \, c'a \, . \, bc = 0 \text{.........(1)}.$$

Now let (LAB) denote the condition that L, A, B may meet in a point, so that, A and B being fixed lines, (LAB) = 0 is the equation to the point (A, B). Then $ab = \dfrac{(\text{LAB})}{(\text{LAI})(\text{LBI})} \times$ constant, I being the line at infinity; hence (1) becomes

$$(\text{LAB})(\text{LCA}')(\text{LB}'\text{C}') + (\text{LA}'\text{B}')(\text{LC}'\text{A})(\text{LBC}) = 0 \text{ ....(2)},$$

the tangential equation to a curve of the 3rd class, the envelope of L.

If A, B, C meet in a point, and A′, B′, C′ in another point, then C is of the form $\text{A} + \kappa\text{B}$, and C′ of the form $\text{A}' + \lambda\text{B}'$, so that (2) reduces to the form

$$(\text{LAB})(\text{LA}'\text{B}')[(\text{LCA}') + (\text{LC}'\text{A})] = 0 \text{ ....(3)}.$$

which is the equation to three points.

2. Next, to find the locus of the double points of the system. If f be a double point of the involution aa', bb', cc', we have

$$ab \, . \, fa' \, . \, b'f + a'b' \, . \, fa \, . \, bf = 0 \text{...........................(4)}.$$

Now $\quad fa' =$ (perpendicular from f on A′) cosec $\widehat{\text{L,A}'}$

$$= \frac{\text{A}'}{(\text{LA}'\text{I})} \times \text{constant, suppose;}$$

therefore (4) becomes $\quad (\text{LAB})\,\text{A}'\text{B}' + (\text{LA}'\text{B}')\,\text{AB} = 0$......................(5),

similarly $\quad (\text{LAC})\,\text{A}'\text{C}' + (\text{LA}'\text{C}')\,\text{AC} = 0$......................(6).

For simplicity, we may take $\text{A} = x$, $\text{A}' = y$, and let $\text{L} = \lambda x + \mu y + \nu$, $\text{B} = ax + by + c$, $\text{B}' = a'x + b'y + c'$, $\text{C} = fx + gy + h$, $\text{C}' = f'x + g'y + h'$.

Then $(\text{LAB}) = \begin{vmatrix} \lambda, & \mu, & \nu \\ 1, & 0, & 0 \\ a, & b, & c \end{vmatrix} = -\mu c + \nu b, \quad (\text{LA}'\text{B}') = \begin{vmatrix} \lambda, & \mu, & \nu \\ 0, & 1, & 0 \\ a', & b', & c' \end{vmatrix} = \lambda c' - \nu a',$

$(\text{LAC}) = \begin{vmatrix} \lambda, & \mu, & \nu \\ 1, & 0, & 0 \\ f, & g, & h \end{vmatrix} = -\mu h + \nu g, \quad (\text{LA}'\text{C}') = \begin{vmatrix} \lambda, & \mu, & \nu \\ 0, & 1, & 0 \\ f', & g', & h' \end{vmatrix} = \lambda h' - \nu f',$

(5) becomes $\quad (-\mu c + \nu b)\,y\text{B}' + (\lambda c' - \nu a')\,x\text{B} = 0,$

(6) becomes $\quad (-\mu h + \nu g)\,y\text{C}' + (\lambda h' - \nu f')\,x\text{C} = 0;$

therefore, eliminating λ, μ, ν from these equations and $\lambda x + \mu y + \nu = 0$,

$$\begin{vmatrix} x, & y, & 1 \\ c'x\text{B}, & -cy\text{B}', & by\text{B}' - a'x\text{B} \\ h'x\text{C}, & -hy\text{C}', & gy\text{C}' - f'x\text{C} \end{vmatrix} = 0, \text{ or } \begin{vmatrix} 1, & 1, & 1 \\ c'\text{B}, & -c\text{B}', & by\text{B}' - a'x\text{B} \\ h'\text{C}, & -h\text{C}', & gy\text{C}' - f'x\text{C} \end{vmatrix} = 0 \text{ ..(7)},$$

a cubic passing through the intersection of B, B′; C, C′; A, A′.

3. If A, B, C meet in a point P, $bh = cg$, and if A′, B′, C′ meet in a point Q, $a'h' = c'f'$, and the equation (7) reduces to

$$\text{B}'\text{C}\,[cf'x + bh'y + ch'] = \text{BC}'\,[a'xh + c'gy + c'h],$$

or $$\text{B}'\text{C}ch'\left[\frac{a'x}{c'} + \frac{by}{c} + 1\right] = \text{BC}'c'h\left[\frac{a'x}{c'} + \frac{by}{c} + 1\right],$$

therefore $\text{PQ} = 0$, or $ch'\text{B}'\text{C} = c'h\text{BC}'$, a conic round (A, A′), (B, B′), (C, C′), P and Q.

5770. (By the EDITOR.)—If α, β, γ be the sines of the angles of a plane triangle whose other parts are denoted in the usual way (s_1 being put for $s-a$, &c.), Δ its area, and Δ_1 the area of the escribed triangle, that is to say, the triangle whose vertices are the centres of the escribed circles of the original triangle; prove that

$$\Delta_1 : \Delta = 2R : r = abc : 2\,s_1s_2s_3 = 4\alpha\beta\gamma : (\beta+\gamma-\alpha)\,(\gamma+\alpha-\beta)\,(\alpha+\beta-\gamma),$$

and deduce therefrom a Solution of Question 5735.

Solution by the PROPOSER.

By a well known relation we have $\Delta = rs$; also, by BOOTH's *Geometrical Methods*, Vol. II., Art. 216, $\Delta_1 = 2Rs$;

hence
$$\frac{\Delta_1}{\Delta} = \frac{2R}{r} = \frac{abc}{2r\Delta} = \frac{abcs}{2\Delta^2} = \frac{abcs}{2ss_1s_2s_3} = \frac{abc}{2s_1s_2s_3}$$

$$= \frac{4abc}{(b+c-a)\,(c+a-b)\,(a+b-c)} = \frac{4\alpha\beta\gamma}{(\beta+\gamma-\alpha)\,(\gamma+\alpha-\beta)\,(\alpha+\beta-\gamma)} \quad\text{...(1).}$$

From the equation $\dfrac{\Delta_1}{\Delta} = \dfrac{abcs}{2\Delta^2}$, we have $\Delta\Delta_1 = \frac{1}{2}abcs$..................(2).

If O_1, O_2, O_3 be, as usual, the escribed centres, we have

$$\Delta O_1BC = \tfrac{1}{2}ar_1 = \tfrac{1}{2}a\frac{\Delta}{s_1},\ \ \Delta O_2CA = \tfrac{1}{2}br_2 = \tfrac{1}{2}b\frac{\Delta}{s_2},\ \ \Delta O_3AB = \text{\&c.};$$

hence, by adding the areas of these three triangles to the area of the original triangle, we obtain $\Delta_1 = \Delta\left(1 + \dfrac{a}{2s_1} + \dfrac{b}{2s_2} + \dfrac{c}{2s_3}\right)$..............(3),

which, in briefer notation, is the expression used by Professor COCHEZ (with the phrase, "TODHUNTER a démontré, &c.") as the basis of his solution of Quest. 5735, published in our October number. It is, however, merely an unsolved exercise, certainly undemonstrated, given in Chapter xvi. of TODHUNTER's *Trigonometry*.

Applying (1) to the solution of the particular case that forms Quest. 5735, we have $\alpha = 1$, $\beta = \frac{1}{2}\sqrt{3}$, $\gamma = \frac{1}{2}$; $\dfrac{\Delta_1}{\Delta} = \dfrac{4}{\sqrt{3}-1} = 2(\sqrt{3}+1)$.

5775. (By W. R. ROBERTS, M.A.)—From each vertex of the triangle formed by the three inflexional tangents to a cubic can be drawn a pair of tangents; show that the three pairs all touch the same conic.

Solution by Professor TOWNSEND, F.R.S.

The equation of the cubic, referred to the three inflexional tangents, being of the form $(x+y+z)^3 + 6mxyz = 0$............................(1),

where $x+y+z=0$ is that of the inflexional axis, and the six tangents from any point to a cubic passing through the six intersections with it of the polar conic of the point, the three pairs of tangents in question pass consequently through the three pairs of single intersections with (1) of the three conics

$$(x+y+z)^2+2myz=0,\quad (x+y+z)^2+2mzx=0 \quad \text{.........}(2, 3),$$
$$(x+y+z)^2+2mxy=0 \quad \text{..........................}(4);$$

eliminating, therefore, x, y, z, respectively, between (2), (3), (4), and (1), we get for their equations

$$9(y+z)^2+8myz=0,\; 9(z+x)^2+8mzx=0,\; 9(x+y)^2+8mxy=0 \;\text{...}(5, 6, 7),$$

which shew that they intersect the three opposite sides of the triangle of reference in three pairs of points on the conic

$$9(x+y+z)^2+8m(yz+zx+xy)=0 \quad \text{.....................}(8),$$

and are in consequence three pairs of tangents to the reciprocal conic

$$(2m+9)(x+y+z)^2-(8m+27)(yz+zx+xy)=0 \quad \text{...........}(9);$$

which is the pretty property in question.

The two reciprocal conics (8) and (9) to the imaginary conic $x^2+y^2+z^2=0$, to which the tangential triangle of reference is self-reciprocal, having, as is evident from the form of their equations, double contact with each other, and with the two reciprocal conics

$$yz+zx+xy=0 \quad\text{and}\quad (x+y+z)^2-4(yz+zx+xy)=0,$$

circumscribed and inscribed respectively to the tangential triangle, at the two points of intersection, real or imaginary, of the whole four with the axis of inflexion $x+y+z=0$; and that line having in consequence a common pole $x=y=z$ with respect to the four, which point is also its harmonic pole with respect to the tangential triangle as well as to the cubic itself; hence, when the inflexional axis is at infinity, and when the tangential becomes in consequence the asymptotic triangle of the cubic, the four conics are at once concentric, coaxal, and similar; their common centre and any pair of conjugate axes of form being the centre and a pair of conjugate axes of inertia of the area of the triangle, and all four being circles concentric with each other and with the triangle when the latter is equilateral, a configuration to which the entire system in its most general form may, it is evident, be always reduced by projection.

5771. (By the Rev. W. A. WHITWORTH, M.A.)—If c_n denote the number of combinations of $2n$ things taken n at a time, and $c_0=1$, prove that

$$u_1=c_0c_n+c_1c_{n-1}+c_2c_{n-2}+\ldots+c_nc_0=2^{2n},$$
$$u_2=\frac{c_0c_n}{1}+\frac{c_1c_{n-1}}{2}+\frac{c_2c_{n-2}}{3}+\ldots+\frac{c_nc_0}{n+1}=c_n\frac{2n+1}{n+1}.$$

Solution by the Rev. D. THOMAS, M.A.; Prof. FICKLIN; *and others.*

Now $$c_n=\frac{2n!}{(n!)^2}=2^n\frac{1.3.5\ldots(2n-1)}{n!};$$

therefore $\quad c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots = (1-4x)^{-\frac{1}{2}} = u_1'$,

and $\quad c_0x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \ldots + \frac{c_n}{n+1}x^{n+1} + \ldots = \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}} = u_2'$;

therefore $\quad u_1 =$ coefficient of x^n in $(u_1')^2$

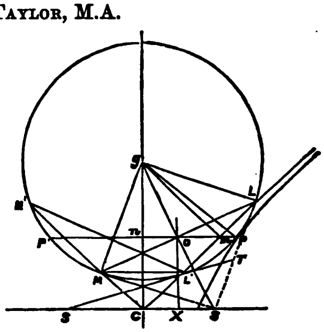
$=$ coefficient of x^n in $(1-4x)^{-1} = 2^{2n}$,

$u_2 =$ coefficient of x^{n+1} in $u_1'u_2'$

$=$ coefficient of x^{n+1} in $\frac{1}{2}(1-4x)^{-\frac{1}{2}} = \frac{c_{n+1}}{2} = \frac{2n+1}{n+1}c_n$.

5212. (By Professor Wolstenholme, M.A.)—A circle is drawn touching both branches of a fixed hyperbola in P, P′, and meeting the asymptotes in L, L′, M, M′: prove that (1) LL′ = MM′ = major axis; (2) the tangents at L, M meet in one focus, and those at L′, M′ in the other, and the angle between either pair is constant, supplementary to the angle between the asymptotes; (3) the directrices bisect LM, L′M′; (4) PP′ bisects LL′, MM′, LM, L′M′; (5) the tangents at L, L′ intersect on a rectangular hyperbola passing through the foci and having one of its asymptotes coincident with MM′ (because $\angle$ CSL + $\angle$ CS′L′ = angle between the asymptotes); (6) LM, L′M′ touch parabolas having their foci at the foci of the hyperbola, and the tangents at their vertices the directrices of the hyperbola.

Solution by C. Taylor, M.A.



Let the normal at P meet the conjugate axis in g, which will be the centre of the circle. Let n be the projection of P on that axis, and let Pn meet LL′ in m.

4. By a property of the normal,

$$Cn \cdot ng = \frac{CA^2}{CB^2} \cdot Cn^2 = mn^2;$$

therefore gm is at right angles to and bisects the chord LL′, &c.

1. Hence

$$(\tfrac{1}{2}LL')^2 = Pg^2 - mg^2 = Pn^2 - mn^2 = CA^2.$$

3. Hence the projection of LL upon the axis is equal *to the distance between the directrices*, and it is also equal to the difference of the abscissa of L, L′, or of L, M, and therefore to *twice the abscissa of o, the middle point of* LM, which therefore lies upon a directrix.

6. Since, by a property of the normal, Sg passes through the point of concourse of PP′ with the directrix, which (by § 4 and § 3) is o, the middle point of LM; therefore LM is at right angles to So, and its envelope is a parabola, &c.

2. By a property of the normal, Pg (the radius of the circle) is a mean proportional to Sg and go; and therefore S is the pole of LM, and *the angle* LSM *is supplementary to* twice the supplement of ML′L, or to *twice* LCS.

5. If T be the intersection of SL, S′L′, the difference of the angles at S and S′ in the triangle STS′ (whereof that of S′ is equal to MSS′) is equal to the angle between the asymptotes; and therefore the locus of T is a rectangular hyperbola, of which SS′ is a diameter, and MM′ an asymptote.

[An analytical solution of the problem is given on pp. 29, 30 of Vol. XXIX. of the *Reprint*.]

5751. (By T. Mitcheson, B.A., L.C.P.)—If r, r_1, r_2, r_3 be the radii of the circles of contact of a triangle, shew that

$$a+b+c = 3\,(r^{-1}r_1r_2r_3)^{\frac{1}{2}}-(rr_1^{-1}r_2r_3)^{\frac{1}{2}}-(rr_1r_2^{-1}r_3)^{\frac{1}{2}}-(rr_1r_2r_3^{-1})^{\frac{1}{2}}.$$

Solution by G. G. Storr; J. O'Regan; *and others.*

Using the convenient notation s_1 for $s-a$, &c., we have

$$\frac{r_1r_2r_3}{r} = \frac{s}{\Delta}\cdot\frac{\Delta}{s_1}\cdot\frac{\Delta}{s_2}\cdot\frac{\Delta}{s_3} = \frac{s\Delta^2}{s_1s_2s_3} = s^2, \text{ \&c. \&c.};$$

therefore $\quad 3\,(r^{-1}r_1r_2r_3)^{\frac{1}{2}}-\text{\&c.} = 3s-s_1-s_2-s_3 = a+b+c.$

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150. Note on Question 5569; *by the* Proposer.

I do not quite understand Mr. Mackenzie's objection (*Reprint*, Vol. XXIX., p. 42, Foot-note) to my solution of this question. It might have been objected that $\Sigma A_i \dfrac{dx}{dt_i}$ is not a function of x; but, this admitted, I see no flaw in the solution. Here, at all events, is a simple verification, in which the suppositions, alleged by Mr. Mackenzie to be essential, are not introduced. We have identically

$$xy\log xy = x\,.\,y\log y + y\,.\,x\log x,\quad \frac{d}{dt_i}\,.\,xy = x\,.\,\frac{d}{dt_i}\,y + y\,.\,\frac{d}{dt_i}\,x.$$

Multiply these identities by arbitrary constants and add: then, writing

$$\phi(x) \equiv A\,.\,x\log x + \Sigma A_i\frac{d}{dt_i}\,x,\quad \text{we have } \phi(xy) = x\,.\,\phi(y) + y\,.\,(\phi).$$

It seems a matter of no slight interest that, in solutions of a functional equation, the arbitrary constants may in many cases be replaced by opera-

tive symbols—such as $\dfrac{d}{dt}$, &c.—and this even when the constants vanish. For instance, in Professor TOWNSEND's solution (*loc. cit.*) the form of $\phi(x)$ obtained is $\phi(x) = a \, . \, x \log x + cx$, where c, if a mere quantity, must vanish. But if c be replaced by the operator $\Sigma A_i \dfrac{d}{dt_i}$, ϕ satisfies the given equation. We cannot similarly replace a; and it would be interesting to determine the conditions under which such substitutions are or are not permissible. Some other examples will be found in a note in the *Messenger of Mathematics*, Vol. VII., pp. 156, 157.

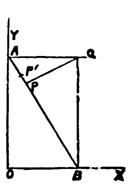
5657. (By S. A. RENSHAW.)—The sides bc, ca, ab of a triangle inscribed in a conic, with focus f, are produced to meet the directrix in k, l, m; fk, fl, fm are joined, and fr, fs, ft are drawn perpendicular to them and meeting the directrix in r, s, t; r, s, t are joined with any point p on the conic, and produced to meet it again in d, e, g; show that ad, be, cg, joined and produced, meet in the same point on the directrix.

Solution by CHRISTINE LADD.

The question reciprocates into the following known proposition:—A, B, C are the sides of a triangle circumscribed about a circle, and R, S, T are lines through the centre perpendicular to the lines joining the centre to the vertices BC, CA, AB respectively. From the intersections of R, S, T with any tangent to the circle the remaining tangents D, E, G are drawn. They cut the sides A, B, C in three points on a diameter of the circle.

5705. (By W. GALLATLY, B.A.) — Prove geometrically that the envelop of a series of coaxial ellipses, the sum of whose semi-axes $= c$, is the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

Solution by the PROPOSER; Prof. SCHEFFER; *and others.*



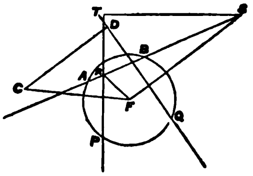
If a rod AB, of length c, slide on two rectangular axes OX, OY; then the series of ellipses are traced out by a series of points P on the rod. Now let P, P′ be two consecutive points, tracing out two consecutive ellipses.

Then, Q being the instantaneous centre, QP, QP′ are normals to the two ellipses at P, P′. Therefore if P is the point of intersection of the ellipses, QP, QP′ coincide; therefore QP is perpendicular to AP.

Therefore P is on the four-cusped hypocycloid enveloped by AB.

5289. (By S. ROBERTS, M.A.)—Show that, if a system of conics having a common focus, envelop a given curve, and have their eccentricities proportional to the focal distances of the poles of their directrices with respect to a circle about the common focus as centre, the locus of the poles is a parallel of the reciprocal of the given curve with respect to the circle.

Solution by the PROPOSER; Rev. J. L. KITCHIN, M.A.; *and others.*



Let AB be a tangent to the given curve, S its point of contact, F the focus of a conic touching AB at S, ABPQ a circle about F as centre. Also, let TP be the directrix of the conic meeting AB in R. Draw ST perpendicular to TP and join FS. Then DQ, the polar of S, is parallel to RF; and if CD is the polar of R, it is also perpendicular to DQ at D the pole of AB. Let C be pole of the directrix, which is on CD, and join CF. Then we have the points S, C and their polars DQ, TP with respect to a circle. Hence $\dfrac{\text{CF}}{\text{CD}} = \dfrac{\text{FS}}{\text{ST}} =$ eccentricity; and if while the position of the line AB varies, the eccentricity is proportional to CF, we have CD constant. The locus of C is therefore a parallel of the locus of D, or the reciprocal of the locus of S.

5789. (By W. J. WRIGHT, Ph.D.)—Prove that the condition of two equal roots in $Ax^3+Bx^2+Cx+D=0$, is

$$4(B^2-3AC)(C^2-3DB)=(BC-9AD)^2.$$

I. *Solution by* Professor COCHEZ.

Désignons par $f(x)$ le premier membre de l'équation, et par $f(x, y)$ ce qu'il devient quand on y remplace x par $\dfrac{x}{y}$, et qu'on effectue, on a

$$f(x, y) = Ax^3+Bx^2y+Cxy^2+Dy^3.$$

Pour que $f(x)$ ait une racine double, on doit avoir

$$f'_x(x, y) = 0, \quad f'_y(x, y) = 0 \text{ avec } y = 1,$$

or $\quad f'_x(x, y) = 3Ax^2+Bxy+Cy^2, \quad f'_y(x, y) = Bx^2+2Cxy+3Dy^2.$

Faisons $y = 1$ et égalons à 0, on a

$$3Ax^2+Bx+C=0, \quad Bx^2+2Cx+3D=0;$$

équations qui doivent avoir une racine commune.

La condition pour que ces équations aient une racine commune est

$$\left(\frac{C}{3A}-\frac{3D}{B}\right)^2+\left(\frac{2B}{3A}+\frac{2C}{B}\right)\left(\frac{6BD}{3AB}-\frac{2C^2}{3AB}\right)=0,$$

ou, après simplifications, $4(B^2-3AC)(C^2-3BD) = (BC-9AD)^2$.

[See also SALMON's *Higher Algebra*, Ed. 3, Arts. 167, 195.]

II. *Solution by* Rev. D. THOMAS, M.A.; Rev. J. L. KITCHIN, M.A.; J. C. GLASHAN, M.A.; *and others.*

If a, b, b are the roots of $Ax^3+Bx^2+Cx+D=0$ ....(1),
b is a root of $3Ax^2+2Bx+C=0$ ....(2),

$a+b, a+b, 2b$ are roots of $Ax^3+2Bx^2+\left(\frac{B^2}{A}+C\right)x+\left(\frac{BC}{A}-D\right)=0$...(3);

also $2b$ is a root of $Ax^3+2Bx^2+4Cx+8D=0$ ....(4),

and $2b$ is a root of $3Ax^2+4Bx+4C=0$ ....(5);

that is, (3), (4), (5) have a common root given by (3)−(4),

or $$x(B^2-3AC)=(9AD-BC).$$

Substituting in (5), we get

$$3A(9AD-BC)^2+4B(9AD-BC)(B^2-3AC)+4C(B^2-3AC)^2=0,$$
$$3A(9AD-BC)^2+4(B^2-3AC)(9ABD-3AC^2)=0.$$

or $$4(B^2-3AC)(C^2-3BD)=(BC-9AD)^2.$$

5774. (By J. J. WALKER, M.A.)—Prove that the vector (ω) of the centre of the circle that passes through the terms of the vectors α, β, γ, is determined by the equation
$$\Sigma\left[(\omega-\alpha)T^2(\beta-\gamma)S(\gamma-\alpha)(\alpha-\beta)\right]=0.$$

I. *Solution by* Professor MINCHIN, M.A.

The centre of the circle circumscribing a triangle ABC is the centre of mean position of its vertices for the multiples sin 2A, sin 2B, sin 2C; or for the multiples $a\cos A$, $b\cos B$, $c\cos C$. Hence

$\omega(a\cos A+b\cos B+c\cos C)=a\cos A.\alpha+\ldots$, or $\Sigma(\omega-\alpha)a\cos A=0$.

Now $bc\cos A=-S(\beta-\alpha)(\gamma-\alpha)$, &c., and $a=T(\beta-\gamma)$;

therefore $\Sigma\left[(\omega-\alpha)T^2(\beta-\gamma)S(\beta-\alpha)(\gamma-\alpha)\right]=0$.

The vectors to the centre of inscribed circle and to the orthocentre can in the same way be written down at once, the latter point being the mean centre for the multiples tan A, tan B, tan C.

II. *Solution by the* PROPOSER.

Assume $$\omega-\alpha+\lambda(\omega-\beta)+\mu(\omega-\gamma)=0,$$
or $$(1+\lambda+\mu)\omega-\alpha-\lambda\beta-\mu\gamma=0;$$

then, multiplying by α, β, γ successively, and equating the scalar parts separately with zero, we have

$$(1+\lambda+\mu)S\alpha\omega-\alpha^2-\lambda S\alpha\beta-\mu S\gamma\alpha=0,$$
$$(1+\lambda+\mu)S\beta\omega-S\alpha\beta-\lambda\beta^2-\mu S\beta\gamma=0,$$
$$(1+\lambda+\mu)S\gamma\omega-S\gamma\alpha-\lambda S\beta\gamma-\mu\gamma^2=0.$$

But $(\omega-\alpha)^2=(\omega-\beta)^2=(\omega-\gamma)^2=-k^2+\omega^2$, say, gives
$$2S\alpha\omega=k^2+\alpha^2,\ 2S\beta\omega=k^2+\beta^2,\ 2S\gamma\omega=k^2+\gamma^2;$$

which values being substituted in the three equations above, and these being solved for λ, μ (and k^2), there results

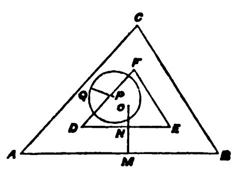
$$\lambda : \mu = (S\alpha\beta + S\beta\gamma - S\gamma\alpha - \beta^2)(2S\gamma\alpha - \gamma^2 - \alpha^2)$$
$$: (S\beta\gamma + S\gamma\alpha - S\alpha\beta - \gamma^2)(2S\alpha\beta - \alpha^2 - \beta^2)$$
$$= S(\alpha-\beta)(\beta-\gamma)T^2(\gamma-\alpha) : S(\beta-\gamma)(\gamma-\alpha)T^2(\alpha-\beta).$$

By symmetry,

$$1 : \lambda = S(\gamma-\alpha)(\alpha-\beta)T^2(\beta-\gamma) : S(\alpha-\beta)(\beta-\gamma)T^2(\gamma-\alpha).$$

5781. (By Elizabeth Blackwood.)—P and Q are two random points within an equilateral triangle; find the chance that the circle of which P is the centre and PQ the radius lies wholly within the equilateral triangle.

Solution by E. B. Seitz.



Let ABC be any triangle, O the centre of its inscribed circle. Draw the triangle DEF, making its sides parallel to those of ABC, and at a distance from them equal to PQ, and draw ONM perpendicular to AB. Now while PQ is given in length and direction, and is less than OM, the area of the triangle DEF represents the number of ways P and Q can be taken, so that the circle whose centre is P and radius PQ, will lie wholly within the triangle ABC.

Let PQ = MN = x, OM = r, the radius of the inscribed circle of ABC, s = the perimeter of ABC, area ABC = Δ, θ = the angle which PQ makes with some fixed line. Then area DEF = $\Delta\left(1-\frac{x}{r}\right)^2$; an element of the circle at Q is $x\,dx\,d\theta$; the limits of x are 0 and r, and those of θ are 0 and 2π. Hence, since the whole number of ways the two points can be taken is Δ^2, the required chance is

$$p = \frac{1}{\Delta^2}\int_0^r\int_0^{2\pi}\Delta\left(1-\frac{x}{r}\right)^2 x\,dx\,d\theta = \frac{2\pi}{\Delta}\int_0^r\left(1-\frac{x}{r}\right) x\,dx$$
$$= \frac{\pi r^2}{6\Delta} = \frac{\pi r}{3s} = \frac{\pi}{18\sqrt{3}},$$

when the triangle is equilateral. The chance is the same, if P and Q are two random points within any circumscriptible polygon. When the polygon becomes a circle, $s = 2\pi r$, and $p = \frac{1}{6}$.

5805. (By C. Leudesdorf, M.A.)—A conic has five-point contact with the curve $y = a + bx + cx^3$ at a point P, and cuts the curve at another

point Q. Shew that, as P moves along the curve, the envelop of the line PQ is $y = a + bx + \frac{343}{100}cx^3$.

Solution by J. HAMMOND, M.A.; T. MORLEY, L.C.P.; *and others.*

Eliminating y between the equations

$$y = a + bx + cx^3, \quad Ax^2 + By^2 + 2Hxy + 2Gx + 2Fy + C = 0 \;\ldots\ldots(1, 2),$$

we obtain an equation of the 6th degree in x of the form

$$Bc^2x^6 + \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0.$$

If the conic (2) has a five-point contact with (1) at a point P, whose abscissa is x_1, and cuts it again in a point Q, whose abscissa is x_2; it is clear that $5x_1 + x_2 = 0$. Also the equation of the line PQ is

$$y - (a + bx_1 + cx_1^3) = (x - x_1)\,\frac{b\,(x_2 - x_1) + c\,(x_2^3 - x_1^3)}{x_2 - x_1}$$

$$= (x - x_1)\left\{b + c\,(x_2^2 + x_1x_2 + x_1^2)\right\} = (x - x_1)(b + 21cx_1^2).$$

Or $$y = a + bx + 21cxx_1^2 - 20cx_1^3 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Differentiating (3) with respect to x, we obtain

$$42xx_1 - 60x_1^2 = 0, \text{ or } x_1 = \frac{7x}{10}.$$

Substituting in (3), the envelop of PQ is $y = a + bx + \frac{343}{100}cx^3$.

5432. (By Professor BALL, LL.D.)—From any point perpendiculars are drawn to the generators of the surface $z\,(x^2 + y^2) - 2mxy = 0$; show that the feet of the perpendiculars lie upon a plane ellipse.

Solution by Prof. NASH, M.A.; Prof. MOREL; *and others.*

The generators of the surface $z\,(x^2 + y^2) = 2mxy$ are given by the equations $z = k$, $k\,(x^2 + y^2) = 2mxy$. Any point on the generator may be represented by the coordinates $\rho\cos\theta$, $\rho\sin\theta$, k, where $k = m\sin 2\theta$. If this be a point on the locus, the line joining it to a given point $\alpha\beta\gamma$ is perpendicular to the generator, therefore

$$\frac{x - \rho\cos\theta}{\alpha - \rho\cos\theta} = \frac{y - \rho\sin\theta}{\beta - \rho\sin\theta} = \frac{z - k}{\gamma - k}$$

is perpendicular to $$\frac{x - \rho\cos\theta}{\cos\theta} = \frac{y - \rho\sin\theta}{\sin\theta} = \frac{z - k}{0};$$

therefore $$(\alpha - \rho\cos\theta)\cos\theta + (\beta - \rho\sin\theta)\sin\theta = 0,$$

therefore $$\rho = \alpha\cos\theta + \beta\sin\theta;$$

therefore, for the point on the locus,

$$x = a\cos^2\theta + \beta\sin\theta\cos\theta,\ \ y = a\sin\theta\cos\theta + \beta\sin^2\theta,\ \ z = m\sin 2\theta;$$

therefore $\tan\theta = \frac{y}{x},\ \ \sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}},\ \ \cos\theta = \frac{x}{(x^2+y^2)^{\frac{1}{2}}};$

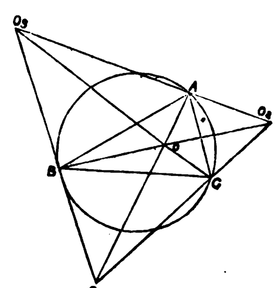
therefore $$x = \frac{x(ax+\beta y)}{x^2+y^2} \text{ or } x^2+y^2 = ax+\beta y \text{ ....(1)},$$

and $$\beta x + ay = a\beta + \frac{a^2+\beta^2}{2}\sin 2\theta = a\beta + \frac{a^2+\beta^2}{2}\,\frac{2}{m} \text{ ....(2)};$$

therefore the locus is the intersection of the plane (2) with the circular cylinder (1).

5510. (By S. Constable.)—If x^2, x_1^2 denote the areas of the inscribed and escribed squares (with respect to the side passing through the angle A) of the triangle formed by joining the escribed centres of a triangle; prove that $x^2 = 4\mathrm{R}s\,\frac{ar_1}{(a+r_1)^2}$, $x_1^2 = 4\mathrm{R}s\,\frac{ar_1}{(a-r_1)^2}$, with similar relations for the other four squares.

Solution by D. Edwardes; J. O'Regan; *and others.*



If R′ be the radius of the circle about $O_1O_2O_3$, we have

$$x = \frac{2\mathrm{R}'\sin O_1 \sin O_2 \sin O_3}{\sin O_1 + \sin O_2 \sin O_3}.$$

But $O_1 = 90° - \frac{1}{2}\mathrm{A}$, &c.; also $\mathrm{R}' = 2\mathrm{R}$, because ABC is the orthocentric triangle of $O_1O_2O_3$; therefore

$$x = \frac{4\mathrm{R}\cos\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}\mathrm{C}}{\cos\frac{1}{2}\mathrm{A} + \cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}\mathrm{C}};$$

and $$\cos\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}\mathrm{B}\cos\tfrac{1}{2}\mathrm{C} = \frac{s}{4\mathrm{R}},$$

$$\cos^2\tfrac{1}{2}\mathrm{A} = \frac{as}{4\mathrm{R}r_1},\ \text{ whence } x^2 = \frac{4as\mathrm{R}r_1}{(a+r_1)^2}.$$

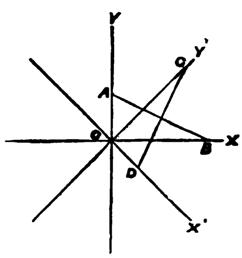
For the escribed square,

$$x_1 = \frac{2\mathrm{R}'\sin O_1 \sin O_2 \sin O_3}{\sin O_2 \sin O_3 - \sin O_1};\ \text{ whence, as before, } x_1^2 = \frac{4as\mathrm{R}r_1}{(a-r_1)^2}.$$

5719. (By R. F. Scott, M.A.)—A straight line of length a slides between two rectangular axes, and a perpendicular is drawn to the line

through a fixed point on it; show that this perpendicular envelopes a parallel of the hypocycloid $(y+x)^{\frac{2}{3}} + (y-x)^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution by J. L. McKenzie, B.A.

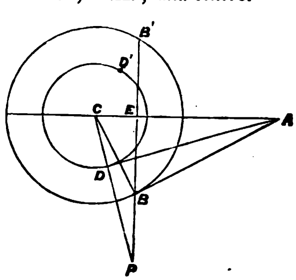


Let AB be the line of length a, sliding between the axes OX, OY; draw CD bisecting AB at right angles, and terminated by OX′, OY′, the bisectors of the angles between the axes. Then it is easily proved by elementary geometry that CD is equal to CB; and CD slides between the rectangular axes OX′, OY′; therefore it envelopes the hypocycloid $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}$, by a well-known property of the four-cusped hypocycloid. (See Todhunter's *Differential Calculus*, 7th edition, p. 363; Gregory's *Examples*, p. 222, &c.). And since a line perpendicular to AB through any other fixed point in it, must be parallel to CD, and at a constant distance from it, the envelop of a line so drawn must be a parallel curve to the envelope of CD.

[It is interesting to notice that the envelop of a perpendicular to AB through the intersection of two of its successive positions is also the four-cusped hypocycloid $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.]

5444. (By C. Taylor, M.A.)—The chord of tangents from O to a conic cuts a conic having the same focus and directrix in O′, and SZ is drawn at right angles to SO′ to meet the directrix. Show that the locus of (ZO, SO′) is a conic whose eccentricity is a third proportional to those of the former two.

Solution by Christine Ladd; Professor Nash, M.A.; *and others.*



Reciprocating with respect to the focus, we get the following theorem:—If pairs of tangents AB . AB′, AD . AD′ be drawn from any point A to two concentric circles, centre C, and BD meet BB′ in P, the locus of P is a circle whose radius is a third proportional to the radii of the given circles.

Let CA meet BB′ in E,

then CP : CE :: CA : CD;

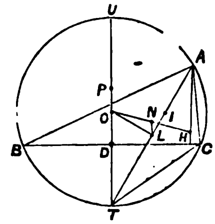
also CE : CB :: CB : CA,

therefore CP : CD :: CB : CD.

151. Geometrical Investigation of the Distance between the Centres of the Inscribed and Nine-Point Circles of any Triangle.

By R. F. Davis, M.A.

Let ABC be a triangle; H its orthocentre, O the centre of its circumscribing circle, and N the middle point of OH, the centre of its nine-point circle. Then, if the arc BC of the circumscribing circle be bisected in T, AT bisects the angle A; and TI being taken equal to TB or TC, I is the centre of the inscribed circle.



Bisect AT in L, then OL is perpendicular to AT. Hence LN is parallel to OT and

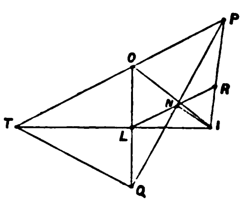
$$= \tfrac{1}{2}(\mathrm{OT} \sim \mathrm{AH}).$$

But $\qquad \mathrm{AH} = 2\mathrm{OD};$

thus $\qquad \mathrm{LN} = \tfrac{1}{2}(\mathrm{TD} \sim \mathrm{OD}) = \tfrac{1}{2}\mathrm{OP},$

if PD = TD. Therefore, if OL be produced to Q, so that QL = OL, then PN passes through Q and PN = QN.

Moreover $\mathrm{TI}^2 = \mathrm{TB}^2 = \mathrm{TD}\,.\,\mathrm{TU} = \mathrm{TO}\,.\,\mathrm{TP}$; hence the triangle TOI is similar to the triangle TIP, and also to the triangle LIR.

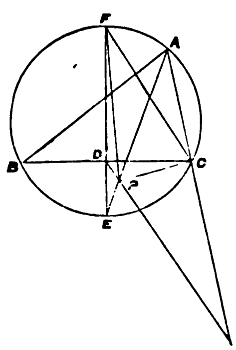


Now, since TI bisects the angle T of the triangle PTQ and $\mathrm{TI}^2 = \mathrm{TP}\,.\,\mathrm{TQ}$, therefore (see Lemma) IN and TP (or NR) are equally inclined to IP. Thus NI = NR; and by similar triangles $\mathrm{NI} : \tfrac{1}{2}\mathrm{IR} = \mathrm{OI} : \mathrm{IL}$. But

$$\mathrm{IR} : \mathrm{IL} = \mathrm{OI} : \mathrm{OT};$$

therefore $\qquad \mathrm{NI}\,.\,\mathrm{OT} = \tfrac{1}{2}\mathrm{OI}^2.$

Lemma. If AP bisect the angle A of a triangle ABC and be a mean proportional between the sides AB, AC; then PD, AC are equally inclined to PC and PD, AB to PB, D being the middle point of BC.



For the triangles ABP, APC are obviously similar, so that the angle $\mathrm{BPC} = \pi - \tfrac{1}{2}\mathrm{A}$.

Hence, if F be the extremity of the diameter of the circumscribing circle through D, FP = FB = FC. Thus $\mathrm{FP}^2 = \mathrm{FD}\,.\,\mathrm{FE}$; and the angle DPF = angle PEF = angle ACF. But FP, FC are equally inclined to PC, wherefore so also are PD, AC.

5559. (By E. B. Seitz.)—A sphere of radius r is intersected by a sphere whose radius is unknown, but less than r; show that the average of the volume common to both spheres is [illegible]πr^3.

Solution by the PROPOSER.

Let x = the unknown radius, y = the distance between the centres of the two spheres, and v = the volume common to both spheres; then

$$v = \tfrac{1}{12}\pi\left[y^3 - 6(r^2+x^2)\,y + 8(r^3+x^3) - \frac{3(r^2-x^2)^2}{y}\right].$$

The limits of x are 0 and r, and those of y are $r-x$ and $r+x$; hence

$$\text{Average} = \int_0^r\int_{r-x}^{r+x} v\,dx\,.\,4\pi y^2 dy \div \int_0^r\int_{r-x}^{r+x} dx\,.\,4\pi y^2 dy$$

$$= \frac{\pi}{14r^4}\int_0^r\int_{r-x}^{r+x}\left[y^5 - 6(r^2+x^2)\,y^3 + 8(r^3+x^3)\,y^2 - 3(r^2-x^2)^2 y\right] dx\,dy$$

$$= \frac{8\pi}{21r^4}\int_0^r (x^6 - 3rx^5 + 3r^2x^4)\,dx = \tfrac{68}{735}\pi r^3.$$

5608. (By E. B. ELLIOTT, M.A.)—An ellipsoid, whose equation referred to its axes (which remain fixed) at any time is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is being deformed by a pure and homogeneous strain in such a way that at each instant $a\dfrac{da}{dt} = b\dfrac{db}{dt} = c\dfrac{dc}{dt} = k^2$; prove that (1) the motion of each point of its surface is at each instant along the normal at it; (2) the velocities, both of different points of the surface at the same time, and of the same point at different times, are inversely as the central perpendiculars on the corresponding tangent planes; and (3) the path of each point is a line of curvature of an hyperboloid.

5765. (By Professor TOWNSEND, F.R.S.)—In the deformation of an hyperboloid of one sheet regarded as a framework of rigid bars moveable freely about their jointed intersections, the centre and axes of the varying surface being supposed to remain fixed, show that—

(a) Every point of it describes, during deformation, the intersection of two quadrics of the system to which it is confocal in every position.

(b) Every line of curvature of it, of either system, generates the quadric of the system, whose intersection with it in any position determines the line.

Solution by E. B. ELLIOTT, M.A.; W. J. C. SHARP, B.A.; *and others.*

(A.) In 5608 the equalities $a\dfrac{da}{dt} = b\dfrac{db}{dt} = c\dfrac{dc}{dt} = k^2$ express that a^2, b^2, c^2 grow at the same constant rate $2k^2$. The result of the deformation is then that the ellipsoid passes in turn through each of the forms of a complete confocal system.

Now consider a point $(a\cos\lambda, b\cos\mu, c\cos\nu)$ on one position. Its velocities are $\frac{dx}{dt} = \frac{da}{dt}\cos\lambda$, $\frac{dy}{dt} = \frac{db}{dt}\cos\mu$, $\frac{dz}{dt} = \frac{dc}{dt}\cos\nu$, since λ, μ, ν do not alter,—the strain being homogeneous,—that is to say, there are
$$\frac{k^2x}{a^2},\ \frac{k^2y}{b^2},\ \frac{k^2z}{c^2}.$$

Hence (1) the direction of the resultant velocity is $\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}$, that of the normal at the point.

Again, (2), its magnitude
$$\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}} \text{ is } k^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}} = \frac{k^2}{p},$$
and so varies inversely as the central perpendicular.

Also, (3), since the curve through any point which cuts a complete series of confocal ellipsoids at right angles is the curve of intersection of the two hyperboloids confocal with the systems which pass through the point, *i.e.*, is a line of curvature on either of those confocal hyperboloids, it follows from (1) that the path of each point is such a line of curvature.

Moreover, (4), since each point on a line of curvature of the varying ellipsoid traces out such a line of curvature on the confocal hyperboloid which passes through that line, the whole line traces out the whole of that confocal.

(B.) The above have been stated merely for the deformation of an ellipsoid; but it requires only a few verbal interchanges to deduce like theorems for either species of central quadric. Professor TOWNSEND's results (1), (2) of Question 5765 are merely the forms that (3), (4) above take for the case of the hyperboloid of one sheet. It remains, then, merely to show that the deformation which he supposes is of the kind considered.

Let a, b, ic', (where $i = \sqrt{-1}$,) be the semi-axes of the hyperboloid, and consider the portions of generators intercepted between the plane of ab and one parallel to it at a distance c'. Then the lengths of the longest and shortest of these intercepts are $(a^2+c'^2)^{\frac{1}{2}}$, $(b^2+c'^2)^{\frac{1}{2}}$ respectively. Now, the generators being supposed rigid, these lengths remain unaltered in the deformation. Therefore
$$a\frac{da}{dt} + c'\frac{dc'}{dt} = 0, \text{ and } b\frac{db}{dt} + c'\frac{dc'}{dt} = 0,$$
that is to say, $a\frac{da}{dt} = b\frac{db}{dt} = -c'\frac{dc'}{dt} = c\frac{dc}{dt}$. Thus the proof is complete.

[Prof. TOWNSEND remarks that both the results in his Question 5765 are obvious, and immediate consequences from the known property that, in every two or more positions of the varying surface, the two or more corresponding positions of any point of it are "corresponding points" of the two or more surfaces, that is, points whose rectangular coordinates parallel to the axes are respectively proportional to the lengths of the axes to which they are parallel.]

5744. (By the Rev. W. A. WHITWORTH, M.A.)—If n men and their wives go over a bridge in single file in random order, subject only to the

condition that there are to be never more men than women gone over, prove that the chance that no man goes over before his wife is $(n+1)\,2^{-n}$.

Solution by the Rev. J. L. Kitchin, M.A.

Take n letters $(a, c, e, \ldots)$, $(b, d, f, \ldots)$ to represent the n men and the n women respectively. Now write them $abcdef\ldots$, where f goes over before e, d before c, &c. Take abc; b may go as in order, or it may act two ways in which women are first. Again, take bcd, and we get the same; or this and the first orders 2^2 ways for women before men.

Proceeding in this way, we get 2^n ways in which women go over before the men, or more women are over than men.

Now take ab, man and wife, and they must keep this order. There are $(n-1)$ sets of pairs left, and they may go over first or last of these sets; therefore they can go over in this order in $n+1$ ways; therefore the chance of this order is that stated in the question.

[Mr. Whitworth states that he cannot understand the foregoing solution, whereon he makes the following criticisms:—

"Mr. Kitchin seems to make the number of ways in which never more men than women are over to be 2^n, whereas it ought to be $\dfrac{2n\,!}{n+1}$. He seems also to make the number of ways in which no man precedes his wife to be $n+1$. It ought to be $\dfrac{2n\,!}{2^n}$. The ratio of the two numbers is obviously the same in either case, $=(n+1)\div 2^n$." To make the matter more definite, Mr. Whitworth adds the following:—

Illustration.—If $n=3$ let A, B, C be the men, and a, b, c their wives. There are $6\,!\div(3+1) = 180$ ways in which they can go over with never more men than women, viz.,

36 such as abcABC	36 such as abABcC
36 such as abAcBC	36 such as aAbBcC,
36 such as aAbcBC	

the 36 of each type being obtained by permuting the abc in six ways and the ABC in six ways. But the number of ways in which no man goes over before his wife is $6\,!\div 2^3 = 90$, viz.,

36 such as abcABC	similarly	12 such as aAbcBC
24 such as abAcBC		12 such as abABcC
		6 such as aAbBcC,

(for when abc are arranged in any order in the 1st, 2nd, and 4th places, the 3rd place can be filled in two ways, and the 5th and 6th letters can be placed in two ways) and the chance is $\frac{90}{180} = \frac{1}{2}$.

In reply, Mr. Kitchin states that he has not taken all the possible ways, as Mr. Whitworth seems to think: that he simply took a specified order; counted up, on that supposition, what would be the result; and found what he has stated. He adds that his result may be erroneous, but that he would like to see his mistake clearly exhibited; though he freely admits that, on questions of probability, he is quite willing to become the pupil of his old pupil, Mr. Whitworth.]

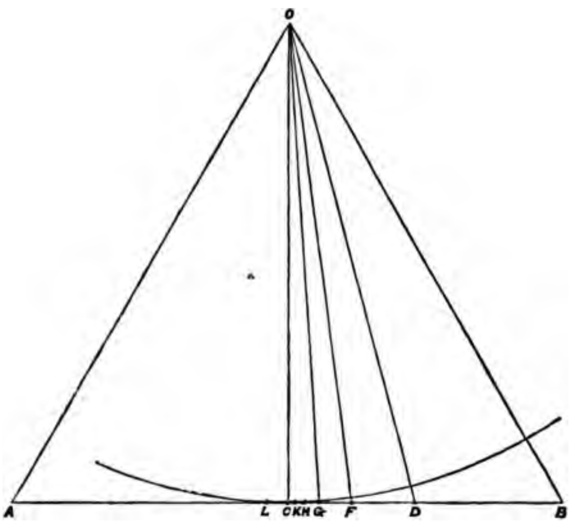
152. On the value of π. *By* C. W. Bourne, M.A.

As circle squarers are still to be found who maintain that the value of π is $3\frac{1}{7}$, it is convenient to have an easy mode of direct refutation.

The following only involves a knowledge of Euclid I. 47, and VI. 3.

To prove that, if we take a circle of diameter 1, the perimeter of the polygon of 192 sides regularly circumscribed to this circle is less than $3\frac{1}{7}$, and that *à fortiori* the circumference of the circle is less than $3\frac{1}{7}$.

Let AB be a side of a regular hexagon circumscribed to the circle; then AOB is an equilateral triangle, and therefore $\angle$ COB = 30°. Bisect



this angle by OD, therefore $\angle$COD = 15°; bisect by OF, therefore $\angle$COF = $7\frac{1}{2}$°; bisect by OG, therefore $\angle$COG = $3\frac{3}{4}$°; bisect by OH, therefore $\angle$COH = $1\frac{7}{8}$°; bisect by OK, therefore $\angle$COK = $\frac{15}{16}$°; make angle COL = $\angle$COK, and let OL meet AB in L; then angle COL = $\frac{15}{16}$; therefore angle LOK = $1\frac{7}{8}° = \frac{360}{192}$; hence LK is a side of regular polygon of 192 sides circumscribed to the circle.

Now, if diameter of circle = 1, OC = ·5.

$$\text{But} \qquad OB^2 = OC^2 + CB^2;\ \text{also } OB = AB = 2CB;$$

$$\text{therefore} \qquad 4CB^2 = OC^2 + CB^2,\ \text{therefore } 3CB^2 = OC^2 = \cdot 25,$$

$$\text{therefore} \qquad CB^2 = \cdot 083333\ldots,\ \text{therefore } CB = \cdot 2886751\ldots,$$

$$\text{therefore} \qquad OB = \cdot 5773503\ldots$$

But, (Euclid VI. 3) CD : DB :: OC : OB :: ·5 : ·5773503...

$$\text{hence} \qquad CD = \cdot 1339746\ldots,\ \text{and } DB = \cdot 1547005\ldots$$

$$\text{Also} \qquad OD^2 = OC^2 + CD^2 = \cdot 25 + \cdot 0179491\ldots = \cdot 2679491\ldots,$$

$$\text{therefore} \qquad OD = \cdot 5176381\ldots$$

Again $CF : FD :: OC : OD :: \cdot5 : \cdot5176381...,$

therefore $CF = \cdot0658262...,$ and $FD = \cdot0681484...$

Also $OF^2 = OC^2 + CF^2 = \cdot25 + \cdot0043330... = \cdot2543330...,$

therefore $OF = \cdot5043145...$

Again $CG : GF :: OC : OF :: \cdot5 : \cdot5043145...,$

therefore $CG = \cdot0327717...,$ and $GF = \cdot0330545...$

Also $OG^2 = OC^2 + CG^2 = \cdot25 + \cdot0010739... = \cdot2510739...$

therefore $OG = \cdot5010728...$

Again $CH : HG :: OC : OG :: \cdot5 : \cdot5010728...$

therefore $CH = \cdot0163683...,$ and $HG = \cdot0164034...$

Also $OH^2 = OC^2 + CH^2 = \cdot25 + \cdot0002679... = \cdot2502679...,$

therefore $OH = \cdot5002678...$

Again $CK : KH :: OC : OH :: \cdot5 : \cdot5002678...,$

therefore $CK = \cdot0081819...,$ and $KH = \cdot0081864...$

Therefore $LK = \cdot0163638...$, and therefore perimeter $= 3\cdot1418496...$, which is less than $3\frac{1}{7}$.

5025. (By the Rev. J. Blissard, B.A.)—Prove that

$$\frac{\Gamma(mx)}{\Gamma(nx)} \text{ (when } x = 0) = \frac{n}{m} \quad \text{.........................(1)},$$

$$\frac{\Gamma(x)}{\Gamma(2x)} - \frac{\Gamma(x+2)}{\Gamma(2x+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(x+4)}{\Gamma(2x+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0 \text{ .... (2)}.$$

Solution by J. Hammond, M.A.; L. W. Jones, B.A.; *and others.*

1. We have $\dfrac{mx\,\Gamma(mx)}{nx\,\Gamma(nx)} = \dfrac{\Gamma(mx+1)}{\Gamma(nx+1)} = 1$, when $x = 0$; therefore &c.

2. The series may be written

$$\frac{1}{\Gamma(x)} \left\{ B(x, x) - B(x+2, x) \frac{\pi^2}{1.2} + B(x+4, x) \frac{\pi^4}{1.2.3.4} - \&c. \right\}$$

$$= \frac{1}{\Gamma(x)} \int_0^1 (1-u)^{x-1} u^{x-1} \left\{ 1 - \frac{\pi^2 u^2}{1.2} + \frac{\pi^4 u^4}{1.2.3.4} - \&c. \right\} du$$

$$= \frac{1}{\Gamma(x)} \int_0^1 (1-u)^{x-1} u^{x-1} \cos \pi u \,.\, du$$

$$= \frac{1}{\Gamma(x)} \int_{-\frac{1}{2}}^{+\frac{1}{2}} (\tfrac{1}{4} - y^2)^{x-1} \sin \pi y \,.\, dy \quad [\text{putting } u = \tfrac{1}{2} - y]$$

$= 0$ [since each term of the integral between limits 0 and $\frac{1}{2}$ is destroyed by the corresponding term of the integral between limits $-\frac{1}{2}$ and 0].

5794. (By Professor TOWNSEND, F.R.S.) — The free motion of a material particle under the action of gravity being supposed disturbed by the resistance of a medium retarding it according to any law; show, from the nature of the case, that the instantaneous parabola decreases in magnitude and descends in position throughout the entire motion, while its axis regredes during the ascent and progredes during the descent of the particle. Determine also, on elementary principles, the rates of production of the several aforesaid changes, in terms of the several particulars of the motion, for any position of the particle.

I. *Solution by* Professor MINCHIN, M.A.

In the undisturbed trajectory, if h is the height of the directrix, v the resultant velocity, and y the height of the particle from the ground,

$$v^2 = 2g(h-y) \qquad\qquad (1).$$

Assume this form for v^2 in the disturbed orbit, and make h variable. Then h will be the height of the directrix of the instantaneous parabola.

Now let R be the resistance of the medium, which acts in the tangent to the path; and the equation of motion will be

$$\frac{dv^2}{ds} = -2g\frac{dy}{ds} - 2\mathrm{R}.$$

Hence (1) gives $$\frac{dh}{ds} = -\frac{\mathrm{R}}{g}, \text{ or } \frac{dh}{dt} = -\frac{\mathrm{R}v}{g} \qquad\qquad (2),$$

which shows that the directrix continually descends during the whole motion and gives the rate of this descent.

Again, the horizontal component of R is $\mathrm{R}\frac{dx}{ds}$, or $\frac{\mathrm{R}v_x}{v}$, where v_x is the horizontal component of the velocity.

Therefore $$\frac{dv_x}{dt} = -\frac{\mathrm{R}v_x}{v} \qquad\qquad (3).$$

Now if m is $\frac{1}{4}$ of the latus rectum,

$$m = \frac{v_x^2}{2g}, \text{ therefore } \frac{dm}{dt} = -\frac{\mathrm{R}v_x^2}{gv} \qquad\qquad (4);$$

which shows that the latus rectum continually diminishes throughout the whole motion. If z = height of vertex above the ground, $z = h - m$.

Hence $$\frac{dz}{dt} = \frac{dh}{dt} - \frac{dm}{dt} = -\frac{\mathrm{R}v_y^2}{gv} \qquad\qquad (5),$$

where v_y is the vertical velocity. Hence the vertex continually descends during the whole motion, and at the rate given by this equation.

Finally, the well-known expression for the range shows that the distance of the particle from the axis at any time is $\frac{v_x \cdot v_y}{g}$. Hence, if x is the abscissa of the particle measured from the original point of projection, the distance ξ of the axis from this point is given by the equation

$$\xi = x + \frac{v_x v_y}{g}.$$

Therefore $$\frac{d\xi}{dt} = v_x + \frac{1}{g}\left(v_x\frac{dv_y}{dt} + v_y\frac{dv_x}{dt}\right).$$

Substituting in this the values $-\frac{Rv_x}{v}$ and $-g-\frac{Rv_y}{v}$ for $\frac{dv_x}{dt}$ and $\frac{dv_y}{dt}$ respectively, we have
$$\frac{d\xi}{dt} = -\frac{2Rv_x \cdot v_y}{gv} \quad\text{..............................(6)},$$
which proves the last of the elegant properties in question, and shows that in the ascent (v_x and v_y being both positive) the axis moves towards the point of projection, and that in the descent (v_y being negative and v_x still positive) the axis moves away from this point.

II. *Solution by the* PROPOSER.

The tendency of the resistance all through the motion, whatever be the law of its action, being to make the velocity of the particle, after description of any elementary arc PQ of its trajectory, less at the far extremity Q than it would have been, all other circumstances being the same at the near extremity P, had gravity been the only force acting during the description of the arc; the instantaneous parabola at Q has consequently, all through the motion, a lower directrix than that at P; and accordingly, touching as it does the latter at Q, which point is consequently the centre of similitude of the two similar and similarly placed parabolas, lies altogether within it in position, and is linearly less than it in magnitude in the ratio of the distances of Q from the lowered and original directrices respectively; while, for the same reason, the vector QV, from Q to the vertex V of the old parabola, diminished in the same ratio, gives the vertex V′ of the new parabola. From these obvious results the several statements in the question are evident and immediate consequences.

To express the rates of production of the several aforesaid changes, in terms of the several particulars of the motion supposed all given or known, for any point P of the trajectory of the particle.—Denoting by x and y the horizontal and vertical coordinates of P to any fixed origin in the plane of the motion, counted positive in the directions of the horizontal velocity of the particle and of gravity, by ξ and η those of the vertex V of the instantaneous parabola, by m the modulus of the parabola, by h the vertical distance of P below its directrix, by v the velocity of the particle, by θ the angle between the directions of its motion and of gravity, and by f the resistance of the medium; then since, during the motion of the particle from P to Q through the elementary arc PQ of its trajectory, $vdv = gdy$ in the instantaneous parabola, and $= gdy - fds$ in the actual trajectory; therefore, if dh be the elementary interval through which the directrix of the former is lowered during the passage from P to Q by the resistance of the medium, we have $gdh = fds$, and therefore
$$\frac{dh}{dt} = \frac{f}{g}\frac{ds}{dt} = \frac{f}{g}v = \frac{f}{g}(2gh)^{\frac{1}{2}} \quad\text{....................(1)},$$
which accordingly is the formula for the rate of depression of the directrix of the instantaneous parabola at the point P of the trajectory.

Again, the action of f being entirely tangential throughout the motion, and affecting in consequence directly the velocity only, but not the direction at any point of the trajectory, since from the geometry of the parabola $h = m\operatorname{cosec}^2\theta$, therefore at once, from (1),
$$\frac{dm}{dt} = \frac{f}{g}(2gh)^{\frac{1}{2}}\operatorname{cosec}\theta \quad\text{..........................(2)},$$

which accordingly is the formula for the rate of diminution of the modulus of the instantaneous parabola at any point P of the trajectory.

And since again, from the same, $d\xi = 2dh \sin\theta\cos\theta$, and $d\eta = dh\sin^2\theta$, therefore again, at once for the same reason, from (1),

$$\frac{d\xi}{dt} = 2\frac{f}{g}(2gm)^{\frac{1}{2}}\cos\theta \quad\text{.........................}\quad (3),$$

and

$$\frac{d\eta}{dt} = \frac{f}{g}(2gm)^{\frac{1}{2}}\sin\theta \quad\text{.........................}\quad (4)$$

which accordingly are the formulæ for the rates of advance and depression of the vertex of the instantaneous parabola at any point P of the trajectory.

As $\cos\theta$ changes sign, while $\sin\theta$ retains its sign, on the passage of the particle through the apsidal points of the trajectory; it follows from the preceding formulæ, as shewn above from the nature of the case, that, of the preceding rates of change, $\frac{d\xi}{dx}$ alone changes sign on the transition from the ascending to the descending motion of the particle.

5800. (By the Editor.)—Sum the series

$$\sin x - \tan^2\tfrac{1}{2}\alpha\sin 3x + \tan^4\tfrac{1}{2}\alpha\sin 5x - \ldots,$$

and deduce therefrom, by integration, a solution of Question 5701.

I. *Solution by* R. Rawson; Prof. Cochez; *and others.*

In De Morgan's *Calculus*, p. 243, and in Todhunter's *Trigonometry*, p. 235, it is proved that

$$\frac{\sin x}{1-2c\cos x+c^2} = \sin x + c\sin 2x + c^2\sin 3x + \ldots \quad\text{.........}(1).$$

In this equation change c to $-c$, then we have

$$\frac{\sin x}{1+2c\cos x+c^2} = \sin x - c\sin 2x + c^2\sin 3x - \ldots \quad\text{.........}(2).$$

Adding (1) to (2), we obtain

$$\frac{1+c^2}{(1-c^2)^2}\cdot\frac{\sin x}{1+\dfrac{4c^2}{(1-c^2)^2}\sin^2 x} = \sin x + c^2\sin 3x + c^4\sin 5x + \ldots \quad\ldots(3).$$

In Williamson's *Integral Calculus*, page 148, it is proved that

$$\int_0^\infty \frac{\sin nx}{x}\,.\,dx = \tfrac{1}{2}\pi;\ \text{hence}\ \int_0^\infty \frac{\sin x}{1+\dfrac{4c^2}{(1-c^2)^2}\sin^2 x}\,.\,\frac{dx}{x} = \tfrac{1}{2}\pi\,.\,\frac{1-c^2}{1+c^2}\quad\ldots(4),$$

which is a more general form of the integral Quest. 5701.

The series in the question, and the integral in Question 5701, are obtained from (3) by putting $c^2 = -\tan^2\frac{1}{2}\alpha$.

In WILLIAMSON's *Integral Calculus*, page 163, and in De MORGAN's *Calculus*, page 631, it is proved that

$$\int_0^\infty \frac{\sin rx}{x^{n+1}} \, . \, dx = -r^n \Gamma(-n) \sin\left(\frac{n\pi}{2}\right) = \mathrm{P}r^n \text{ .... } (5);$$

hence (3) becomes

$$\frac{1+c^2}{(1-c^2)^2}\int_0^\infty \frac{\sin x}{1+\dfrac{4c^2}{(1-c^2)^2}\sin^2 x} \, . \, \frac{dx}{x^{n+1}} = \mathrm{P}\left\{1+3^n c^2+5^n c^4+\ldots\right\} = \mathrm{PS}_n \text{ ...}(6).$$

But
$$\int \mathrm{S}^n \, dc = c\left\{1+3^{n-1}c^2+5^{n-1}c^4+\ldots\right\} = c\mathrm{S}_{n-1};$$

therefore
$$\mathrm{S}_n = \frac{d}{dc}\,(c\mathrm{S}_{n-1}) \text{ .... } (7).$$

From (7), S_n can be readily calculated by observing that $\mathrm{S}_0 = \dfrac{1}{1-c^2}$.

II. *Solution by* J. HAMMOND, M.A.; Professor DARBOUX; *and others.*

Writing $\sin nx = \dfrac{z^n - z^{-n}}{2i}$, and putting, as usual, i for $\sqrt{(-1)}$, the series in the question becomes

$$\frac{z}{2i}\left\{1-z^2\tan^2\tfrac{1}{2}a+z^4\tan^4\tfrac{1}{2}a-\&\text{c.}\right\}-\frac{z^{-1}}{2i}\left\{1-z^{-2}\tan^2\tfrac{1}{2}a+z^{-4}\tan^4\tfrac{1}{2}a-\&\text{c}\right\}.$$

$$= \frac{1}{2i}\left\{\frac{z}{1+z^2\tan^2\frac{1}{2}a}-\frac{z^{-1}}{1+z^{-2}\tan^2\frac{1}{2}a}\right\}$$

$$= \frac{z-z^{-1}}{2i}\left\{\frac{1-\tan^2\frac{1}{2}a}{1+\tan^4\frac{1}{2}a+(z^2+z^{-2})\tan^2\frac{1}{2}a}\right\}$$

$$= \sin x\left\{\frac{1-\tan^2\frac{1}{2}a}{(1+\tan^2\frac{1}{2}a)^2+(z-z^{-1})^2\tan^2\frac{1}{2}a}\right\}$$

$$= \frac{\sin x\cos^2\frac{1}{2}a\cos a}{1-4\sin^2 x\sin^2\frac{1}{2}a\cos^2\frac{1}{2}a} = \cos^2\tfrac{1}{2}a\cos a\,\frac{\sin x}{1-\sin^2 x\sin^2 a}.$$

Now $\displaystyle\int_0^\infty \sin rx \, . \, \frac{dx}{x} = \tfrac{1}{2}\pi$; therefore, if S denote the series, we have

$$\int_0^\infty \mathrm{S} \, . \, \frac{dx}{x} = \tfrac{1}{2}\pi\,(1-\tan^2\tfrac{1}{2}a+\tan^4\tfrac{1}{2}a-\&\text{c.}) = \tfrac{1}{2}\pi\cos^2\tfrac{1}{2}a$$

$$= \cos^2\tfrac{1}{2}a\cos a\int_0^\infty \frac{\sin x}{1-\sin^2 x\sin^2 a} \, . \, \frac{dx}{x}.$$

Thus
$$\int_0^\infty \frac{\sin x}{1-\sin^2 x\sin^2 a} \, . \, \frac{dx}{x} = \tfrac{1}{2}\pi\sec a.$$

III. *Solution by the* REV. J. L. KITCHIN, M.A.; Prof. EVANS, M.A.; *and others.*

Let S denote the sum of the series, and, as usual, put i for $\sqrt{(-1)}$, then

$$\mathrm{S} = \frac{1}{2i}\left\{\epsilon^{ix}-\epsilon^{-ix}-\tan^2\tfrac{1}{2}a\,(\epsilon^{3ix}-\epsilon^{-3ix})+\ldots\ldots\right\}$$

$$= \frac{1}{2i}\left\{\epsilon^{ix}(1-\tan^2\tfrac{1}{2}\alpha\,\epsilon^{2ix}+\tan^4\tfrac{1}{2}\alpha\,\epsilon^{4ix}-\&c.)\right.$$
$$\left. -\epsilon^{-ix}(1-\tan^2\tfrac{1}{2}\alpha\,\epsilon^{-2ix}+\&c.)\right\}$$
$$= \frac{1}{2i}\left\{\frac{\epsilon^{ix}}{1+\tan^2\frac{1}{2}\alpha\,\epsilon^{2ix}}-\frac{\epsilon^{-ix}}{1+\tan^2\frac{1}{2}\alpha\,\epsilon^{-2ix}}\right\}$$
$$= \frac{1}{2i}\left\{\frac{\epsilon^{ix}-\epsilon^{-ix}-\tan^2\frac{1}{2}\alpha\,(\epsilon^{ix}-\epsilon^{-ix})}{1+\tan^4\frac{1}{2}\alpha+\tan^2\frac{1}{2}\alpha\,(\epsilon^{2ix}+\epsilon^{-2ix})}\right\}$$
$$= \frac{\sin x\,(1-\tan^2\frac{1}{2}\alpha)}{1+\tan^4\frac{1}{2}\alpha+2\tan^2\frac{1}{2}\alpha\cos 2x} = \frac{\cos^4\frac{1}{2}\alpha\,(1-\tan^2\frac{1}{2}\alpha)\sin x}{1-\sin^2\alpha\sin^2 x}.$$

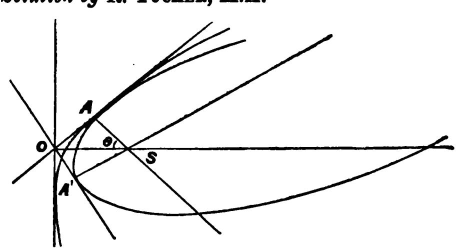
Now $$\int_0^\infty S\,\frac{dx}{x} = \int_0^\infty \sin x\,\frac{dx}{x} - \int_0^\infty \tan^2\tfrac{1}{2}\alpha\sin 3x\,\frac{d3x}{3x} + \&c.$$
$$= \tfrac{1}{2}\pi\left\{1-\tan^2\tfrac{1}{2}\alpha+\tan^4\tfrac{1}{2}\alpha-\&c.\dots\right\} = \tfrac{1}{2}\pi\,\frac{1}{1+\tan^2\frac{1}{2}\alpha};$$

$$\therefore\quad \cos^4\tfrac{1}{2}\alpha\,(1-\tan^2\tfrac{1}{2}\alpha)\int_0^\infty \frac{\sin x}{1-\sin^2\alpha\sin^2 x}\,\frac{dx}{x} = \tfrac{1}{2}\pi\,\frac{1}{1+\tan^2\frac{1}{2}\alpha}$$
$$= \tfrac{1}{2}\pi\cos^2\tfrac{1}{2}\alpha;$$

finally therefore $$\int_0^\infty \frac{\sin x}{1-\sin^2\alpha\sin^2 x}\,\frac{dx}{x} = \tfrac{1}{2}\pi\sec\alpha.$$

5483. (By Rev. W. Roberts, M.A.)—Two parabolas intersect, which are both touched by a given straight line, and which have a given point for focus; find the locus of their intersection when the angle included between their axes is constant.

I. *Solution by* R. Tucker, M.A.



Let SA ($=c$) be the perpendicular from S on given tangent; also let OA, OA′ be the tangents at the vertices A, A′ of two of the parabolas, and $\angle$ ASA′ $= \alpha$. Then the directrices of A, A′ are given by the equations

$$y\sin\theta - x\cos\theta - c\cos\theta = 0,\quad y\sin(\alpha-\theta)+(x+c)\cos(\alpha-\theta) = 0\ \dots\ (1, 2).$$

Hence the parabolas are given by

$$(x-c)^2+y^2 = [y\sin\theta-(x+c)\cos\theta]^2 = [y\sin(\alpha-\theta)+(x+c)\cos(\alpha-\theta)]^2,$$

therefore $$y\sin\theta-(x+c)\cos\theta = [y\sin\alpha+(x+c)\cos\alpha]\cos\theta$$
$$\pm[y\cos\alpha-(x+c)\sin\alpha]\sin\theta,$$

$$\therefore \quad \frac{\sin\theta}{y\sin\frac{1}{2}a+(x+c)\cos\frac{1}{2}a} = \frac{\cos\theta}{y\cos\frac{1}{2}a-(x+c)\sin\frac{1}{2}a} = \frac{\pm 1}{\{y^2+(x+c)^2\}^{\frac{1}{2}}},$$

$$(x^2+y^2+c^2)^2-4c^2x^2 = [y^2+(x+c)^2]^2\sin^2\tfrac{1}{2}a,$$

or $\qquad y^2+(x+c)^2=0, \quad$ or $\quad (x-c)^2+y^2 = [y^2+(x+c)^2]\sin^2\frac{1}{2}a,$

or $$(x^2+y^2+c^2)\cos^2\tfrac{1}{2}a = 2cx\,(1+\sin^2\tfrac{1}{2}a),$$

that is, the circle $\left(x-c\dfrac{1+\sin^2\frac{1}{2}a}{\cos^2\frac{1}{2}a}\right)^2+y^2 = \dfrac{4c^2\sin^2\frac{1}{2}a}{\cos^4\frac{1}{2}a}.$

If $a=\frac{1}{2}\pi$, we have $\qquad (x-3c)^2+y^2 = 8c^2.$

[Mr. TUCKER finds the locus of one of the common points. The other is found by putting $\pi-a$ for a in Mr. TUCKER's equation.]

II. *Solution by* Professor WOLSTENHOLME, M.A.

Taking the given focus for origin of polar coordinates, take $r\cos\theta=a$, the equation of the given line; $r[1+\cos(\theta-\phi)]=2b$, $r[1+\cos(\theta-\phi-a)] = 2c$, the equations of the parabolas; then $b=a\cos\phi$, $c=a\cos(\phi+a)$ are the conditions that they may touch the given line; or at the common point we have $\quad r = \dfrac{2a\cos\phi}{1+\cos(\theta-\phi)} = \dfrac{2a\cos(\phi+a)}{1+\cos(\theta-\phi-a)},$

or ϕ, $(\phi+a)$ are the roots of the equation in z,

$$\cos z\,.\,(2a-r\cos\theta)-\sin z\,.\,r\sin\theta = r.$$

Hence $$\frac{\cos(\phi+\frac{1}{2}a)}{2a-r\cos\theta} = \frac{\sin(\phi+\frac{1}{2}a)}{-r\sin\theta} = \frac{\cos\frac{1}{2}a}{r},$$

whence $$\frac{r^2}{\cos^2\frac{1}{2}a} = (2a-r\cos\theta)^2+r^2\sin^2\theta,$$

or $$r^2\tan^2\tfrac{1}{2}a+4ar\cos\theta-4a^2=0,$$

the equation of a circle, whose centre is at a distance $2a\cot^2\frac{1}{2}a$ from the given point, and $a(1+2\cot^2\frac{1}{2}a)$ from the given straight line, and whose radius is $2a\dfrac{\cos\frac{1}{2}a}{\sin^2\frac{1}{2}a}$. This is, of course, only the locus of *one pair* of their points of intersection; to get the locus of the other pair, we must take, since the same curve is represented by the two polar equations

$$r[1+\cos(\theta-\phi)]=2b, \qquad r[-1+\cos(\theta-\phi)]=2b,$$

$$r = \frac{2a\cos\phi}{-1+\cos(\theta-\phi)} = \frac{2a\cos(\phi+a)}{1+\cos(\theta-\phi-a)},$$

$$\cos\phi\,.\,(2a-r\cos\theta)-\sin\phi\,.\,r\sin\theta = -r,$$

$$\cos(\phi+a)\,.\,(2a-r\cos\theta)-\sin(\phi+a)\,.\,r\sin\theta = r;$$

whence $$\frac{\cos\phi+\cos(\phi+a)}{r\sin\theta} = \frac{\sin\phi+\sin(\phi+a)}{2a-r\cos\theta},$$

$$\frac{\cos(\phi+\frac{1}{2}a)}{r\sin\theta} = \frac{\sin(\phi+\frac{1}{2}a)}{2a-r\cos\theta} = \frac{\cos\phi\sin(\phi+\frac{1}{2}a)-\sin\phi\cos(\phi+\frac{1}{2}a)}{\cos\phi\,.\,(2a-r\cos\theta)-\sin\phi\,.\,r\sin\theta} = \frac{\sin\frac{1}{2}a}{-r},$$

or the locus is $$\frac{r^2}{\sin^2\frac{1}{2}a} = r^2\sin^2\theta+(2a-r\cos\theta)^2,$$

or $$r^2\cot^2\tfrac{1}{2}a+4ar\cos\theta-4a^2=0,$$

(the one found by Mr. TUCKER,) a circle whose centre is at the distance

$2a \tan^2 \frac{1}{2}\alpha$ from the fixed point, and $a(1+2\tan^2\frac{1}{2}\alpha)$ from the given straight line, and of radius $2a\dfrac{\sin\frac{1}{2}\alpha}{\cos^2\frac{1}{2}\alpha}$. The two circles coincide when $\alpha = \frac{1}{2}\pi$. It will be found that the real points lie on the first or second locus according as $\cos\phi\cos(\phi+\alpha)$ is positive or negative.

The reciprocal problem is, Two circles intersect each other at a given angle in two given points, to find the envelope of their common tangents.

Let $x^2+y^2-2\lambda x-a^2=0$, $x^2+y^2-2\mu x-a^2=0$ be the two circles; move the origin to the point $(0, -a)$, and the equations will be

$$x^2+y^2-2\lambda x-2ay = 0, \qquad x^2+y^2-2\mu x-2ay = 0;$$

and, if α be their angle of intersection, $\tan\alpha = \dfrac{(\lambda-\mu)\,a}{\lambda\mu+a^2}$.

But, if $px+qy=1$ be a tangent to both,

$$(1-\lambda p)^2 = (a^2+\lambda^2)(p^2+q^2), \quad (1-\mu p)^2 = (a^2+\mu^2)(p^2+q^2),$$

or λ, μ are the two roots of the equation in z,

$$z^2q^2+2pz+a^2(p^2+q^2)-1 = 0;$$

whence $$(\lambda-\mu)^2 = \frac{4(p^2+q^2)(1-a^2q^2)}{q^4}, \quad \lambda\mu+a^2 = \frac{a^2(p^2+2q^2)-1}{q^2};$$

whence $$\tan^2\alpha = \frac{4(p^2+q^2)(1-a^2q^2)\,a^2}{[a^2(p^2+2q^2)-1]^2},$$

$$4a^4q^4+4a^2q^2(a^2p^2-1)-4a^2p^2+[a^2(p^2+2q^2)-1]^2\tan^2\alpha = 0,$$

$$[a^2(p^2+2q^2)-1]^2(1+\tan^2\alpha) = 4a^2p^2+(a^2p^2-1)^2 = (a^2p^2+1)^2,$$

or $$a^2(p^2+2q^2)-1 = \pm(a^2p^2+1)\cos\alpha,$$

$$a^2p^2(1\pm\cos\alpha)+2a^2q^2 = 1\mp\cos\alpha;$$

and the envelope is

$$\frac{x^2}{a^2\cos^2\frac{1}{2}\alpha}+\frac{y^2}{a^2} = \frac{1}{\sin^2\frac{1}{2}\alpha}, \quad\text{or}\quad \frac{x^2}{a^2\sin^2\frac{1}{2}\alpha}+\frac{y^2}{a^2} = \frac{1}{\cos^2\frac{1}{2}\alpha},$$

two confocal conics (foci $0, \pm a$), whose reciprocals with respect to either of these points are two circles, as before found.

5796. (By Professor Wolstenholme, M.A.—(Prove that

$$\int_1^a F\left(x^2+\frac{a^2}{x^2}\right)\frac{dx}{x} = \int_1^a F\left(x+\frac{a^2}{x}\right)\frac{dx}{x}.$$

I. *Solution by* E. B. Elliott, M.A.; Professor Cochez; *and others.*

In the Solution of Prof. Wolstenholme's Question 4697 (*Reprint*, Vol. XXIV., p. 19) it is shewn that if $S(a, b)$ denote any symmetric function of a and b,

$$\int_1^a S\left(x^2, \frac{a^2}{x^2}\right)\frac{dx}{x} = \int_1^a S\left(x, \frac{a^2}{x}\right)\frac{dx}{x}.$$

And this, since $F(a+b)$ is symmetric in a and b, gives the result required. Indeed the two results are identical for, theoretically, every symmetric function of x, and $\dfrac{a^2}{x}$, is reducible to the form $F\left(x+\dfrac{a^2}{x}\right)$.

II. *Solution by the* PROPOSER.

In the first term put $x^2 = z$, then $\frac{dx}{x} = \frac{1}{2}\frac{dz}{z}$, and the integral becomes

$$\tfrac{1}{2}\int_1^a \mathrm{F}\left(z+\frac{a^2}{z}\right)\frac{dz}{z};$$

in the second put $x^2 = \frac{a^2}{z}$, and the integral becomes

$$-\int_a^1 \mathrm{F}\left(\frac{a^2}{z}+z\right)\frac{dz}{2z}, \quad \text{or } \tfrac{1}{2}\int_1^a \mathrm{F}\left(z+\frac{a^2}{z}\right)\frac{dz}{z},$$

when the sum $= \int_1^a \mathrm{F}\left(z+\frac{a^2}{z}\right)\frac{dz}{z}$, or $\int_1^a \mathrm{F}\left(x+\frac{a^2}{x}\right)\frac{dx}{x}$.

III. *Solution by* R. RAWSON; Rev. D. THOMAS, M.A.; *and others.*

It will be convenient first to prove that

$$\int_1^a \mathrm{F}\left(x^2+\frac{a^2}{x^2}+h\right)\frac{dx}{x} = 2\int_1^{a^{\frac{1}{2}}} \mathrm{F}\left(x^2+\frac{a^2}{x^2}+h\right)\frac{dx}{x}\;\ldots\ldots\ldots(1).$$

By Taylor's and the Binomial theorems, we have

$$\int \mathrm{F}\left(x^2+\frac{a^2}{x^2}+h\right)\frac{dx}{x}$$

$$= \int\left\{\mathrm{F}(h)+\mathrm{F}'(h)\left(x^2+\frac{a^2}{x^2}\right)+\frac{\mathrm{F}''(h)}{1.2}\left(x^2+\frac{a^2}{x^2}\right)^2+\&c.\right\}\frac{dx}{x}\ldots(2),$$

$$\left(x^2+\frac{a^2}{x^2}\right)^n = x^{2n}+\frac{a^{2n}}{x^{2n}}+na^2\left(x^{2n-2}+\frac{a^{2n-2}}{x^{2n-2}}\right)+\frac{n(n-1)}{1.2}\left(x^{2n-4}+\frac{a^{2n-4}}{x^{2n-4}}\right)+\&c.\;\ldots\ldots\ldots(3).$$

From (3), by integration, we obtain

$$\int_1^a \left(x^{2n-2r}+\frac{a^{2n-2r}}{x^{2n-2r}}\right)\frac{dx}{x} = \frac{a^{2n-2r}-1}{n-r} = \int_1^{a^{\frac{1}{2}}}\left(x^{2n-2r}+\frac{a^{2n-2r}}{x^{2n-2r}}\right)\frac{dx}{x}\ldots(4).$$

From (2) and (4) equation (1) is manifest.

Now, $\quad 2\int_1^{a^{\frac{1}{2}}} \mathrm{F}\left(x^2+\frac{a^2}{x^2}+h\right)\frac{dx}{x} = 2\int_1^{a^{\frac{1}{2}}} \mathrm{F}\left(y^2+\frac{a^2}{y^2}+h\right)\frac{dy}{y}.$

Put $y^2 = z^m$, then we have

$$\int_1^{a^{\frac{1}{2}}} \mathrm{F}\left(y^2+\frac{a^2}{y^2}+h\right)\frac{dy}{y} = m\int_1^{a^{\frac{1}{m}}} \mathrm{F}\left(z^m+\frac{a^2}{z^m}+h\right)\frac{dz}{z}$$

$$= m\int_1^{a^{\frac{1}{m}}} \mathrm{F}\left(x \;\; +\frac{a^2}{x^m}+h\right)\frac{dx}{x}$$

Hence $\quad \int_1^a \mathrm{F}\left(x^2+\frac{a^2}{x^2}+h\right)\frac{dx}{x} = m\int_1^{a^{\frac{1}{m}}} \mathrm{F}\left(x^m+\frac{a^2}{x^m}+h\right)\frac{dx}{x}\ldots\ldots(5).$

A particular case of this, viz., $m = 1$, and $h = 0$, becomes the relation in the question.

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